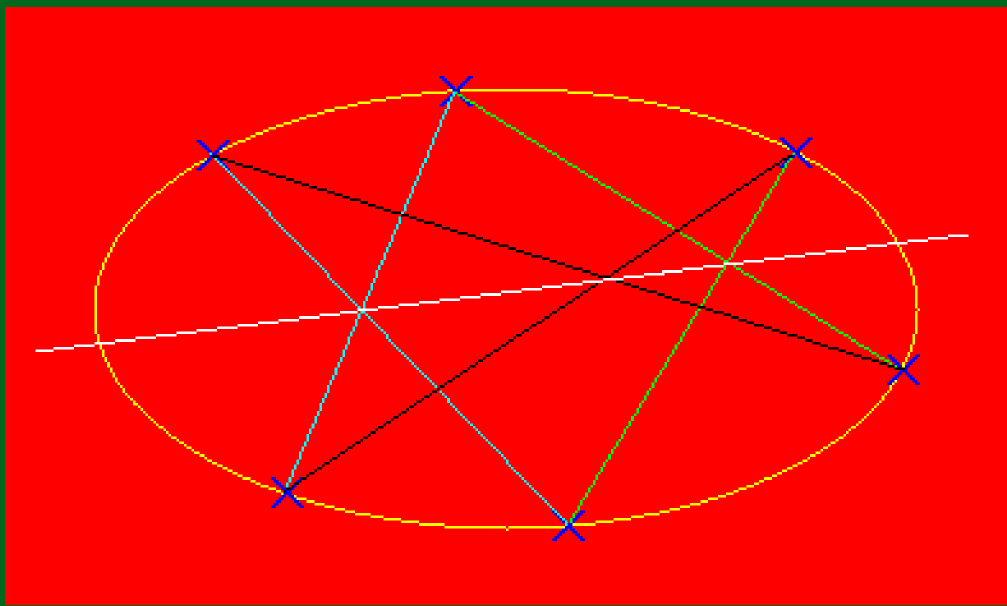
PROJECTIVE GEOMETRY



Projective geometry is a beautiful subject which has some remarkable applications beyond those in standard textbooks. These were pointed to by Rudolf Steiner who sought an exact way of working scientifically with aspects of reality which cannot be described in terms of ordinary physical measurements. His colleague George Adams worked out much of this and pointed the way to some remarkable research done by Lawrence Edwards in recent years. Steiner's spiritual research showed that there is another kind of space in which more subtle aspects of reality such as life processes take place. Adams took his descriptions of how this space is experienced and found a way of specifying it geometrically, which is dealt with in the Counter Space Page.

A brief introduction to the basics of the subject is given in the Basics Page.

See also Britannica: projective geometry

The work of Lawrence Edwards is introduced in the Path Curves Page, and some explorations of his work on further aspects is described in the Pivot Transforms Page. This is mostly pictorial, with reference to documentation.

YOU ARE INVITED TO EXPLORE!

Nick Thomas

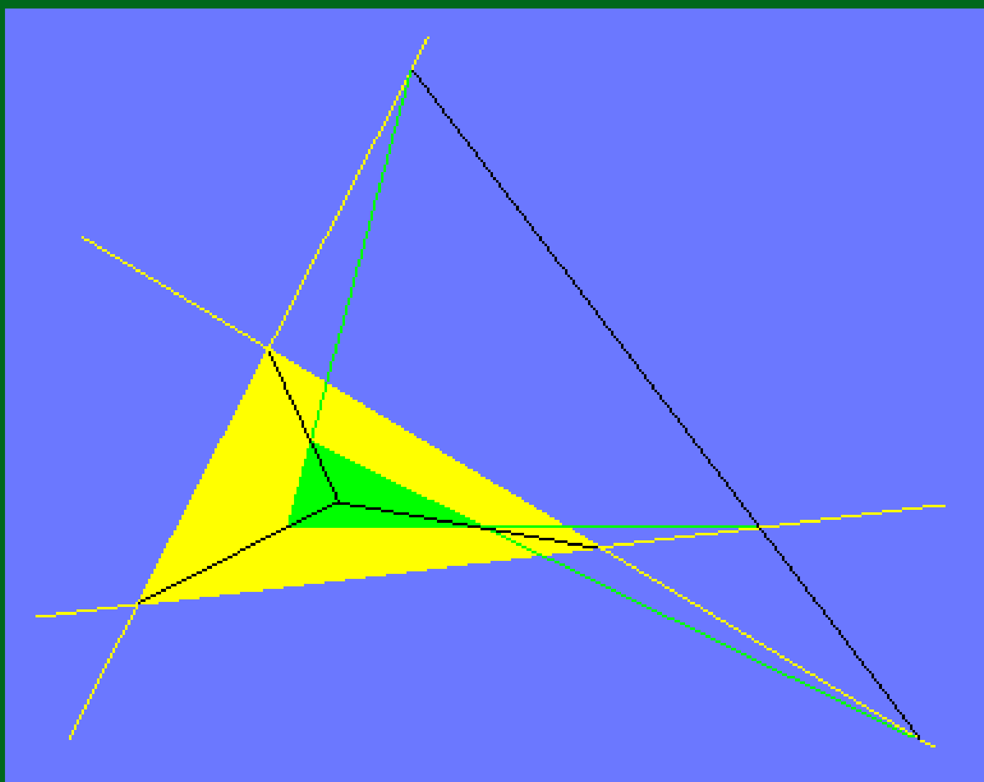
References and selected other sites are listed on the People page.

Feedback welcome! Please include the word "counterspace" in the text and mail to nctsm<At>safe-mail.net, replacing <At> with @ of course.

locations of recent changes

Basics

Projective geometry is concerned with *incidences*, that is, where elements such as lines planes and points either coincide or not. The diagram illustrates DESARGUES THEOREM, which says that if corresponding sides of two triangles meet in three points lying on a straight line, then corresponding vertices lie on three concurrent lines.

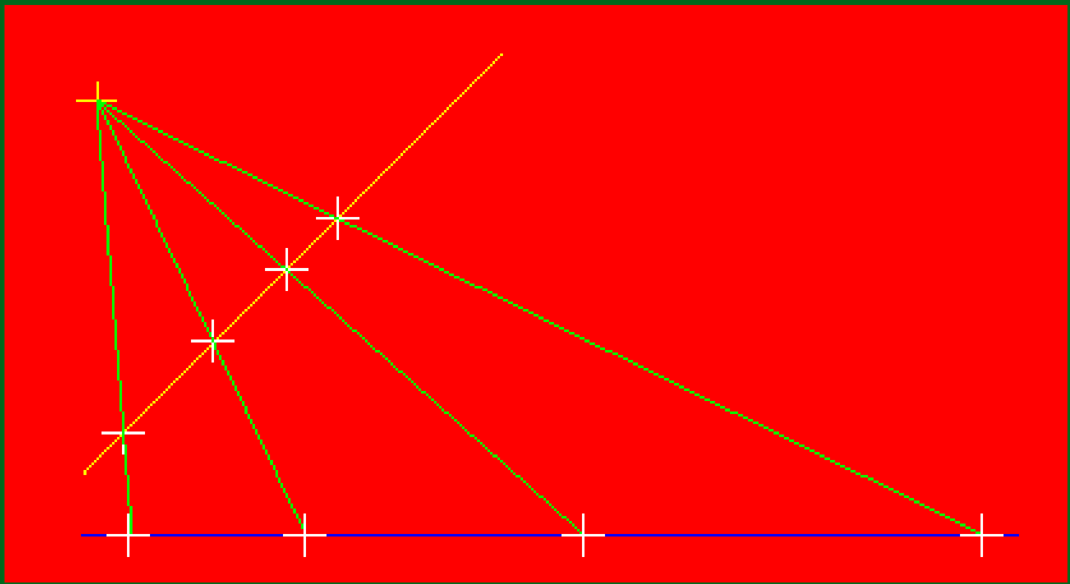


The converse is true i.e. if corresponding vertices lie on concurrent lines then corresponding sides meet in collinear points. This illustrates a fact about incidences and has nothing to say about measurements. This is characteristic of pure projective geometry.

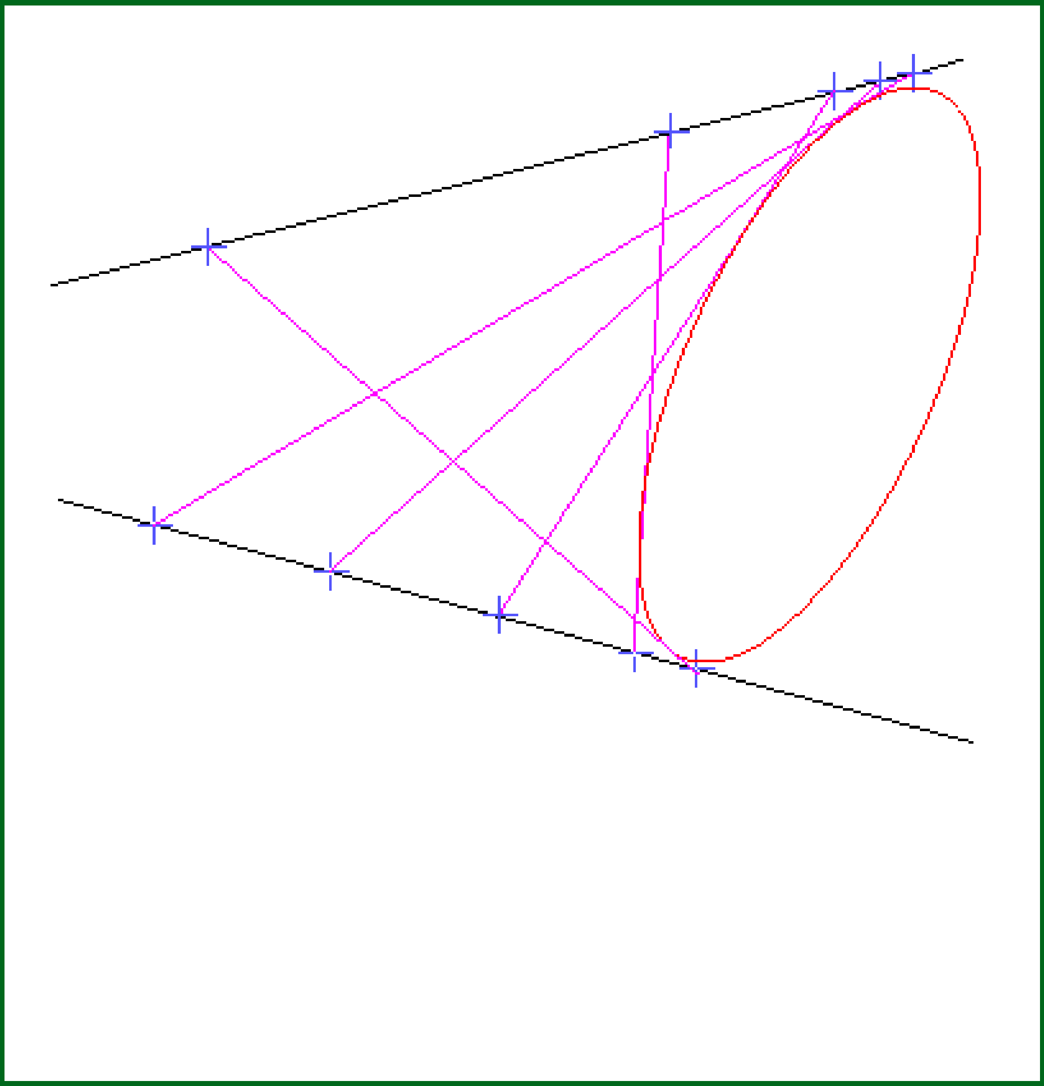
It also illustrates the PRINCIPLE OF DUALITY, for there is a symmetry between the statements about lines and points. If all the words 'point' and 'line' are exchanged in the statement about the sides, and then we replace 'side' with 'vertex', we get the dual statement about the vertices.

The most fundamental fact is that there is one and only one line joining two distinct points in a plane, and dually two lines meet in one and only one point. But what, you may ask, about parallel lines? Projective geometry regards them as meeting in an IDEAL POINT at infinity. There is just one ideal point associated with each direction in the plane, in which all parallel lines in such a direction meet. The sum total of all such ideal points form the IDEAL LINE AT INFINITY.

The next figure shows the process of projection of a RANGE of points on a yellow line into another range on a distinct (blue) line. The set of (green) projecting lines in the point of projection is called a PENCIL of lines. The points are indicated by the centre points of white crosses.



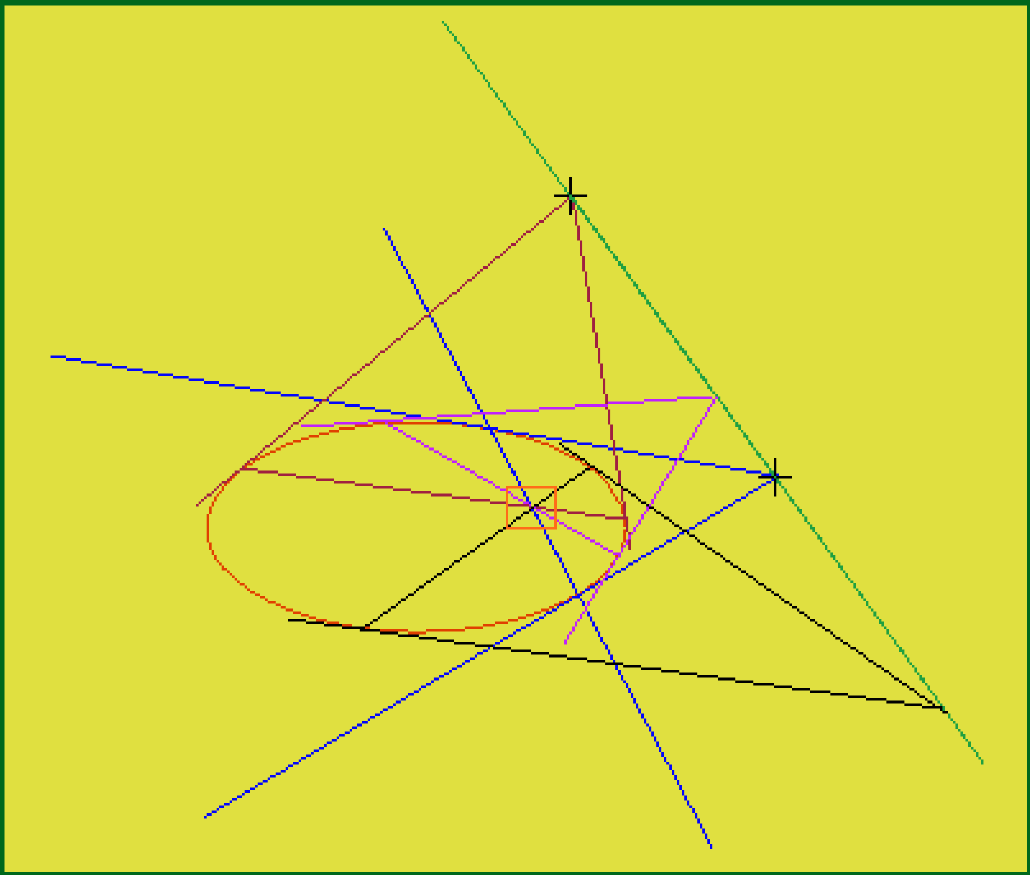
The two ranges are called PERSPECTIVE ranges. The process of intersection of a pencil by a line to produce a range is called SECTION. Projection and section are dual processes. The above procedure may be repeated for a sequence of projections and sections. The first and last range are then referred to as PROJECTIVE RANGES. If corresponding points of two projective ranges are joined the resulting lines do not form a pencil, but instead very beautifully envelope a CONIC SECTION, that is an ellipse, hyperbola or parabola. These are the shapes arising if a plane cuts a cone, and in fact include a pair of straight lines and also, of course, the circle.



Using the dual process a conic can be constructed by points using projective pencils.

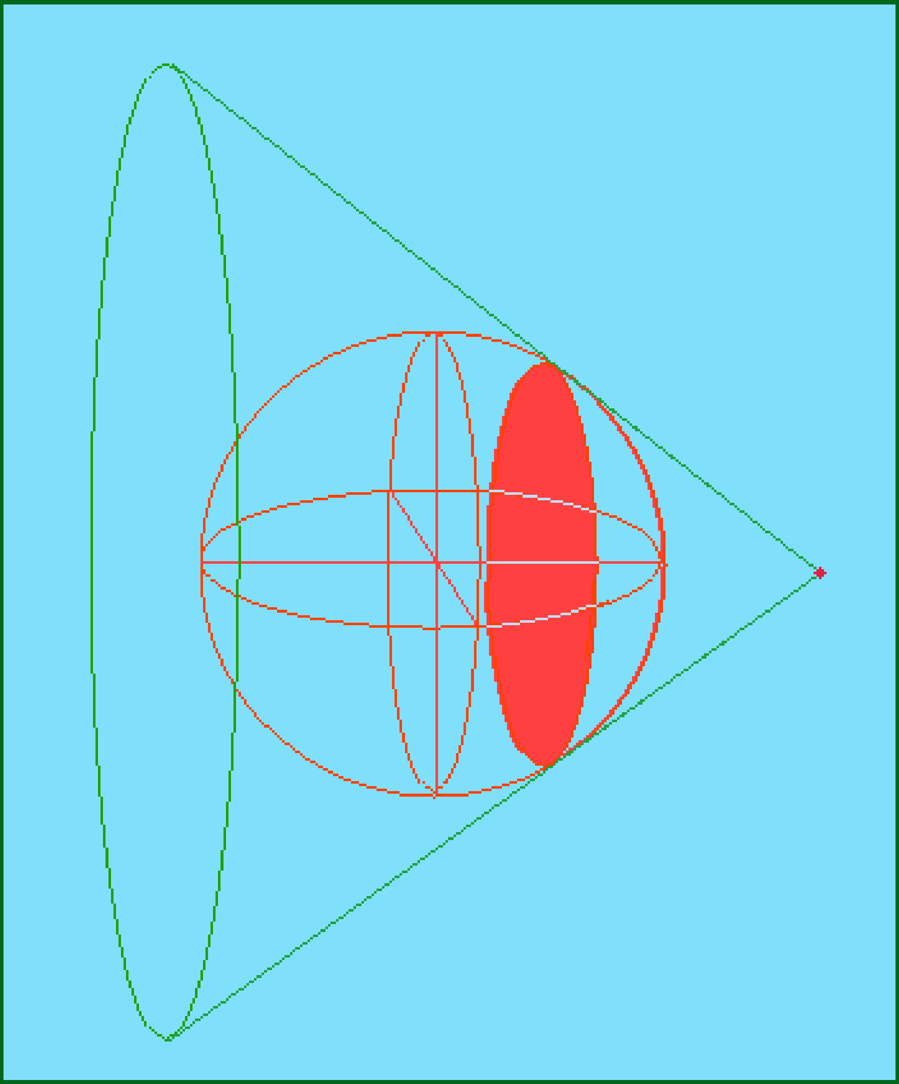
There are many theorems that there is no space to explain here. An example is given on the home page showing Pascal's theorem, and illustrations of others are listed below.

A particularly important subject for counter space is that of polarity, which is related to the principle of duality. If the tangents to a conic through a point are drawn, the line joining the two points of tangency is called the POLAR LINE of the point, and dually the point is called the POLE of that line. This is illustrated below.



The fact to note here is that the polars of the points on a line form a pencil in a point, which is the polar of that line. The situation is self-dual.

In three dimensions we illustrate the same principle but with a sphere and a point. The cone with its apex in that point, and which is tangential to the sphere, determines a plane (red) containing the circle of contact. That plane is the POLAR PLANE of the point, and the point is the POLE of the plane.



Similarly to the two-dimensional case, if we take the polar planes of all the points in a plane, they all contain a common point which is the pole of that plane.   Lines are now self-polar.

When counter space is studied this property of points and planes is used to conceptualise the idea of a negative space, as we reverse the roles of centre and infinity.

AFFINE & METRIC GEOMETRY

Infinity is not invariant for projective geometry, in the sense that ideal points may be transformed by it into other points. In a plane the ideal points form an ideal line, and in space they form an ideal plane or plane at infinity. A special case of projective geometry can be defined which leaves the plane at infinity invariant (as a whole) i.e. ideal elements are never transformed into ones that are not at infinity. This is known as *affine geometry*. A further special case is possible where the volume of objects remains invariant, which is known as *special affine geometry*. Finally a further specialisation ensures that lengths and angles are invariant, which is *metric geometry*, so called because measurements remain unaltered by its transformations.

Other Theorems

- Cross Ratio. The cross ratio of four points is the only numerical invariant of projective geometry (if it can be related to Euclidean space). Flat line pencils and axial pencils of planes containing a common line also have cross ratios.
- Quadrangle Theorem. If two quadrangles have 5 pairs of corresponding sides meeting in collinear points, the sixth pair meet on the same line. Proof indicated using Desargue's Theorem.
- Harmonic Range. Construction of two pairs of points harmonically separated, which have a cross ratio of -1.
- Homology. A basic projective transformation in which corresponding sides meet on a fixed line called the *axis*, and corresponding points lie on lines through the *centre*.
- Pappus' Theorem. This was one of the earliest discoveries, and can be regarded as a special case of Pascal's Theorem.
- Brianchon's Theorem. This is the dual of Pascal's Theorem although it was discovered independently.

Measures and Transformations

It is best to view the first item before those later in the list. They show repeated transformations of the points on a line.

- Breathing (or hyperbolic) Measure. A point is shown moving along a line between two invariant points (with construction).
- Growth Measure with one invariant point at infinity. The ratios of the distances of successive points from the other invariant point are constant.
- Step (or parabolic) Measure, in which the two invariant points coincide. This is how equal steps appear in counter space for our ordinary consciousness.
- Step Measure with both invariant points at infinity, which yields equal steps. The proof follows from the fact that triangles on the same base and between the same parallels are equal in area.
- Circling (or elliptic) Measure in which there are no invariant points. The two auxiliary lines used in the above constructions may be regarded as special cases of a conic.

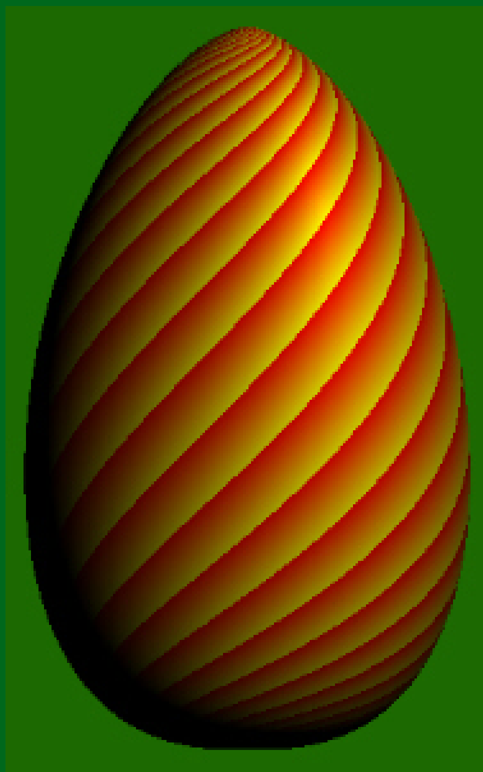
If you attempt to impose three invariant points on a line (e.g. in the first construction by taking the first corresponding pair as coincident) you will find all points are self-corresponding. This is the ***Fixed Point Theorem*** of projective geometry.

The following animations show the application of the above to transformation of a plane, in these examples lines being transformed by means of two measures on two sides of the invariant triangle.

- Projective transformation in which it is demonstrated that parallelism is not conserved.
- Affine transformation where two red parallel lines are transformed into two parallel lines (one green and one blue). This is affine because one side of the invariant triangle is at infinity since each measure has an invariant point at infinity.

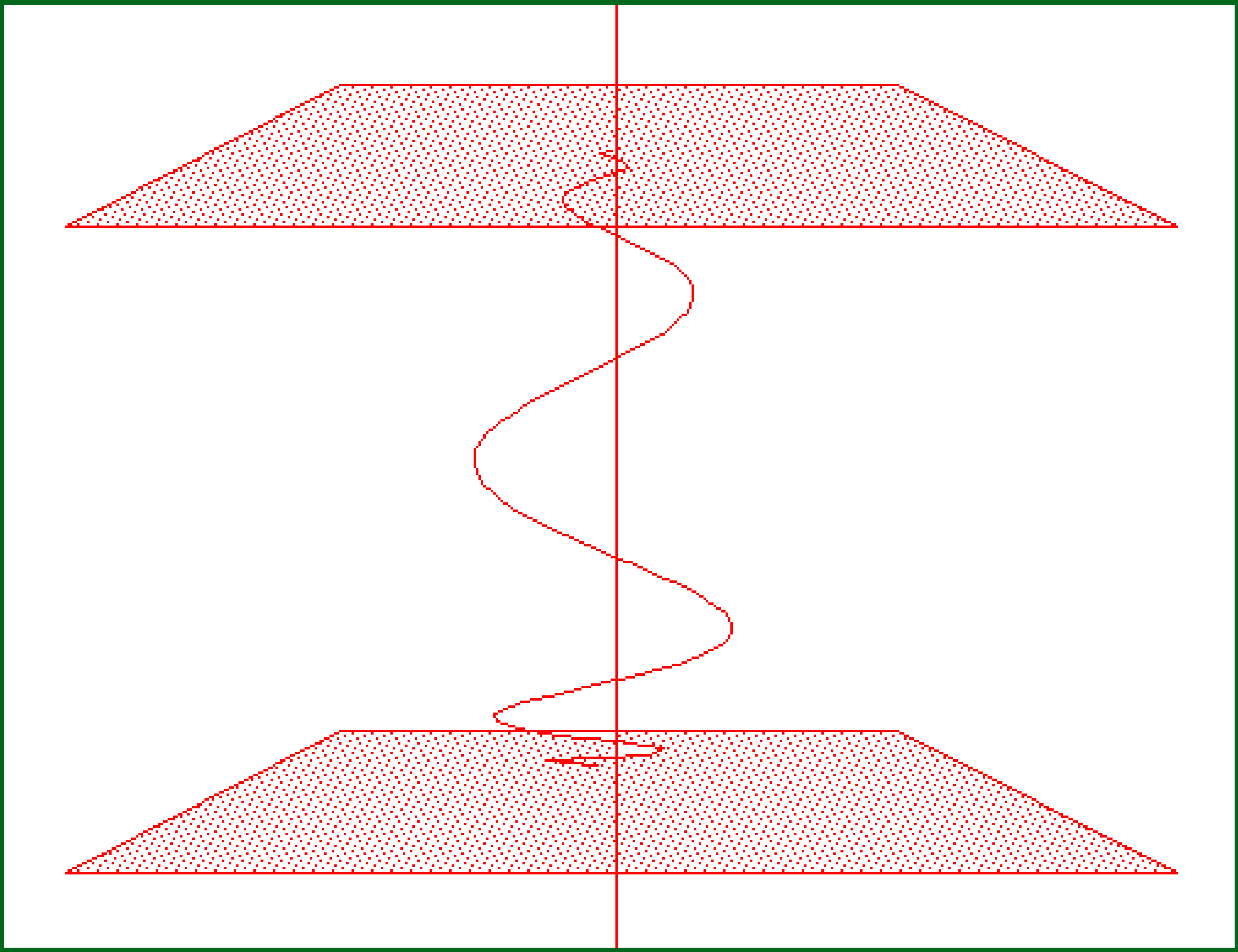
References 6 and 8 and 9 give a good introduction to projective geometry, where the above facts are proved.

Path Curves

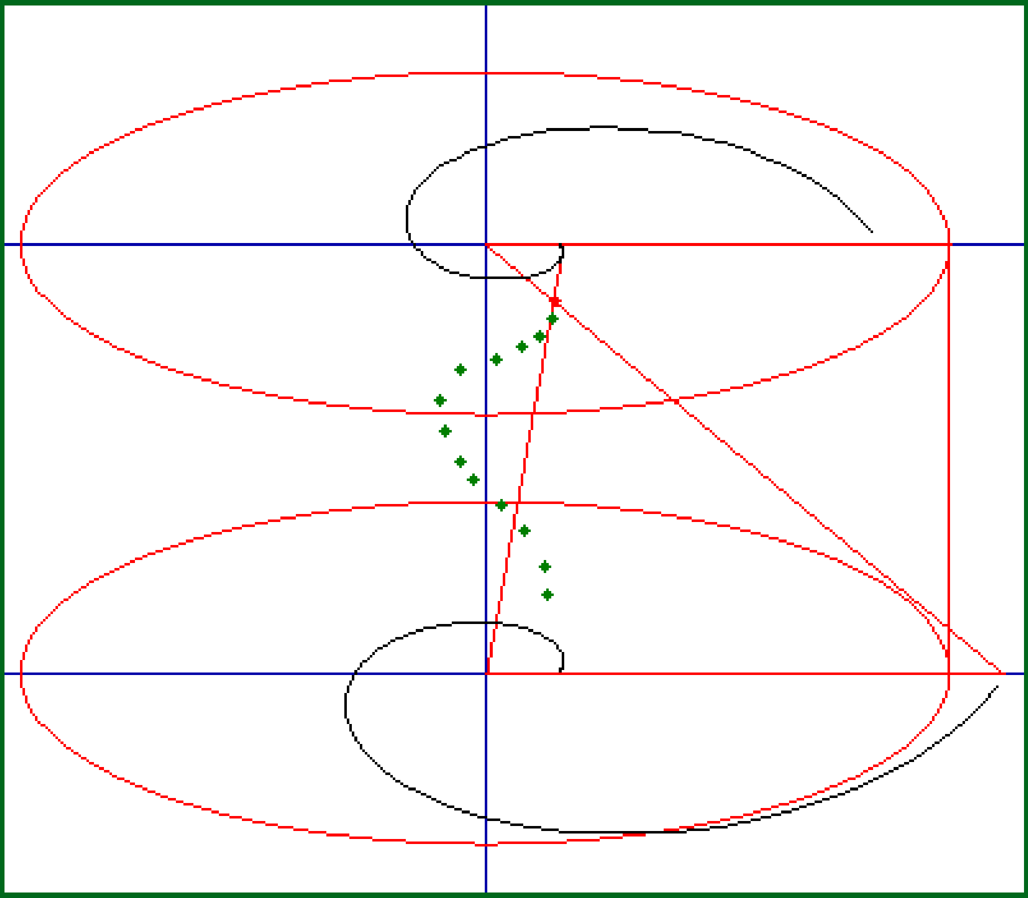


The picture shows an egg form constructed mathematically. The spirals are characteristic of the mathematics and are known as PATH CURVES. They were discovered by Felix Klein in the 19th Century, and are very simple and fundamental mathematically speaking. Geometry studies *transformations* of space, and these curves arise as a result. A simple movement in a fixed direction such as driving along a straight road is an example, where the vehicle is being transformed by what is called a *translation*. In our mathematical imagination we can think of the whole of space being transformed in this way. Another example is rotation about an axis. In both cases there are lines or curves which are themselves unmoved (as a whole) by the transformation : in the second case circles concentric with the axis (round which the points of space are moving), and in the first case all straight lines parallel to the direction of motion. These are simple examples of path curves. More complicated transformations give rise to more interesting curves.

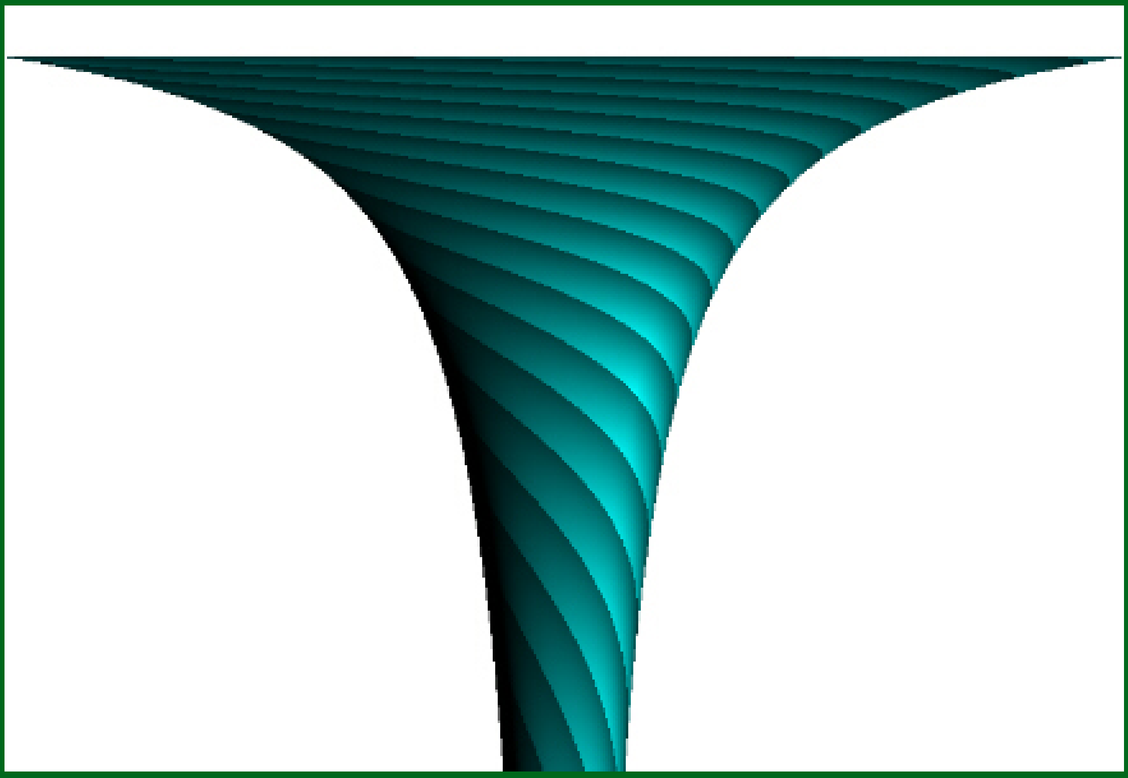
The transformations concerned are projective ones characteristic of projective geometry, which are *linear* because neither straight lines nor planes become curved when moved by them, and incidences are preserved (this is a simplification, but will serve us here). They allow more freedom than simple rotations and translations, in particular incorporating expansion and contraction. Apart from the path curves they leave a tetrahedron invariant in the most general case. George Adams studied these curves as he thought they would provide a way of understanding how space and counter space interact. A particular version he looked at was for a transformation which leaves invariant two parallel planes, the line at infinity where they meet, and an axis orthogonal to them. This is a *plastic* transformation rather than a *rigid* one (like rotation) and a typical path curve together with the invariant planes and axis is shown below.



This will be recognised as the type of curve lying in the surface of the egg at the top of the page. If we take a circle concentric with the axis and all the path curves which pass through it then we get that egg-shaped surface. The construction is shown in the following animation:

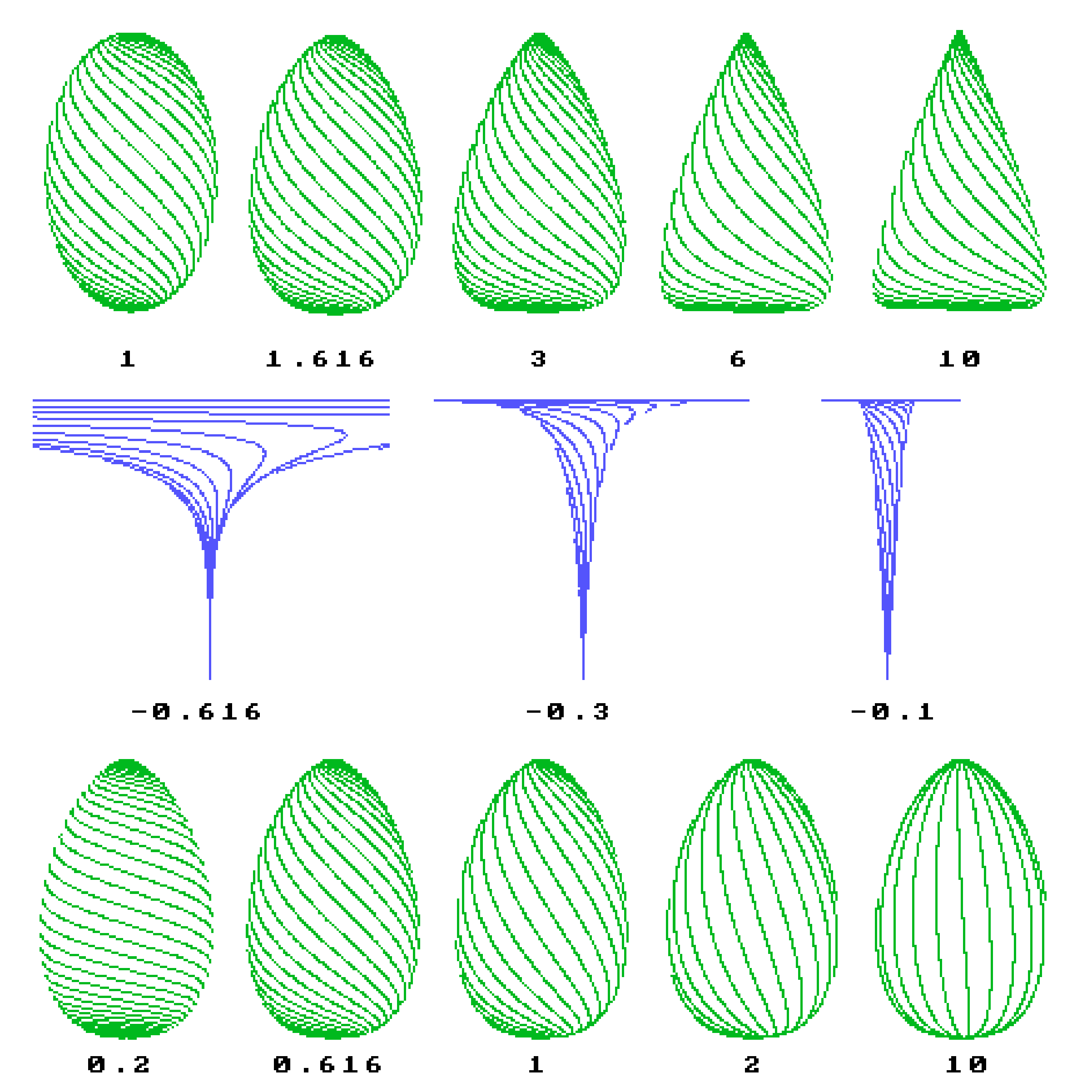


We can vary the transformation to get our eggs more or less sharp, or alternatively we can get vortices such as the following:



In these pictures particular path curves have been highlighted. This particular vortex is an example of a *watery vortex*, so called by Lawrence Edwards because its profile fits real water vortices. It is characterised by the fact that the lower invariant plane is at infinity. If instead the upper plane is at infinity we get what he calls an *airy vortex*.

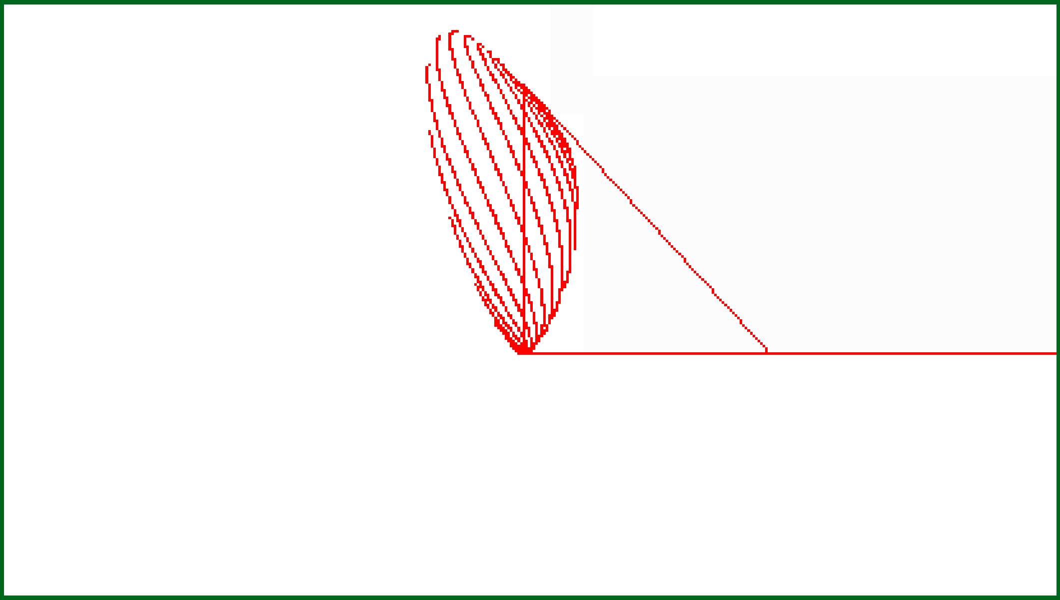
Two parameters are of particular significance: *lambda* and *epsilon*. Lambda controls the shape of the profile while epsilon determines the degree of spiralling. Lambda is positive for eggs and negative for vortices, while the sign of epsilon controls the sense of rotation. This is illustrated below.



The top row shows lambda increasing from 1 (elliptical) to 10. When lambda reaches infinity the form becomes conical. The centre row shows lambda increasing from -0.616 to -0.1 for a vortex. The bottom row shows epsilon varying from 0.2 to

10, and when it reaches infinity the curves are vertical. If it is zero then the path curves become horizontal circles, and strictly speaking the profile is lost.

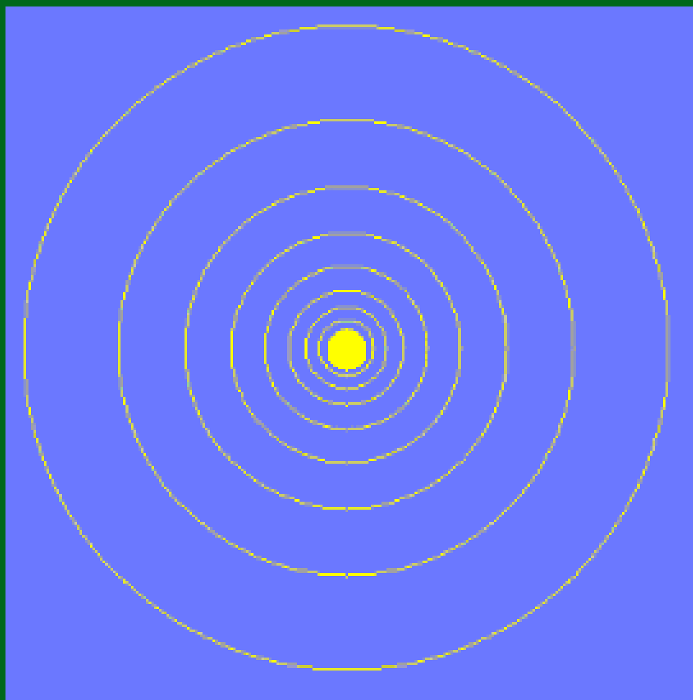
The profile is thus controlled by a single parameter (lambda), and it is scientifically interesting that with such a restriction these curves fit very closely a wide variety of natural forms including eggs, flower and leaf buds, pine cones, the left ventricle of the human heart, the pineal gland, and the uterus during pregnancy. The watery vortex closely fits actual stable water vortices. Together with the airy vortex it also has significance for pivot transforms.   The following shows approximately the way the left ventricle of the heart behaves as a path curve from diastole to systole:



Lawrence Edwards spent many years finding out and testing the above facts experimentally, which he has described in Reference 7. In 1982 he started testing the shapes of the leaf buds of trees through the winter, and found that their lambda value (unexpectedly) varied rhythmically with a period of approximately two weeks. This was his main topic of research in his later years, and the evidence is now very strong - backed by thousands of measurements - that the rhythm corresponds to the conjunctions and oppositions of the Moon and a particular planet for each tree. This is a purely experimental fact and care should be taken in interpreting it.

Download document *Practical Path Curve Calculations* for basic algebra and formulae to work with path curves (Word 97 document).

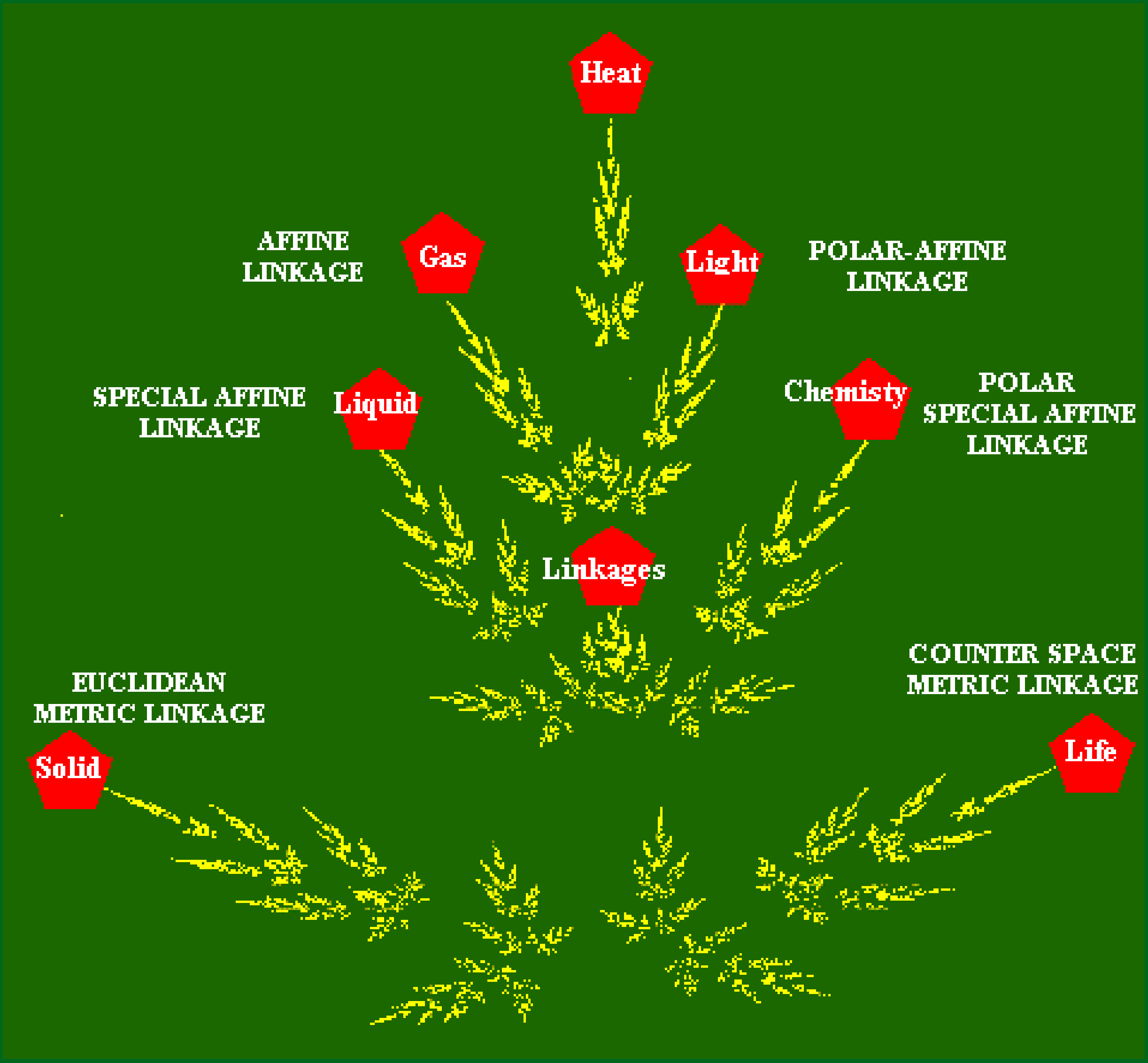
Counter Space



What is Counter Space?

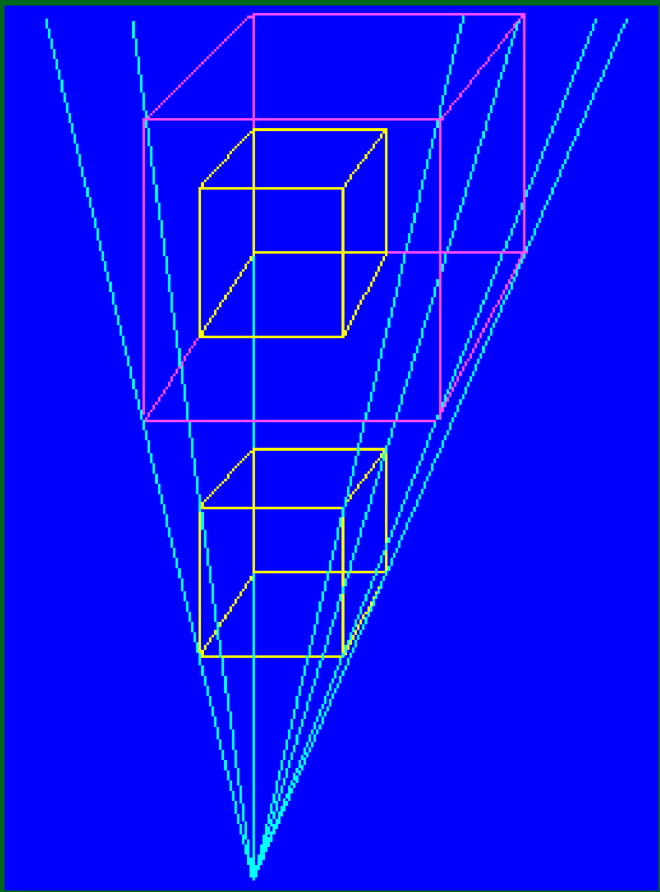
Counter space is the space in which subtle forces work, such as those of life, which are not amenable to ordinary measurement. It is the polar opposite of Euclidean space. It was discovered by the observations of Rudolf Steiner and described geometrically by George Adams and, independently, by Louis Locher-Ernst. Instead of having its ideal elements in a plane at infinity it has them in a "POINT at infinity". They are lines and planes, rather than lines and points as in ordinary space. We call this point the *counter space infinity*, so that a plane incident with it is said to be an *ideal plane* or *plane at infinity* in counter space. It only appears thus for a different kind of consciousness, namely a peripheral one which experiences such a point as an infinite inwardness in contrast to our normal consciousness which experiences an infinite outwardness.

Nick Thomas has explored the idea that objects existing in both spaces at once are subject to strain and stress, and an analysis of these leads to new approaches to gravity and other forces as summarised in the diagram below. The pentagons are 'hot spots' to explore further.



LINKAGES

A linkage is an element that belongs to both Euclidean- and counter-space at once e.g. a point or plane. Suppose a cube is linked to both spaces at once, and is moved upwards away from the inner infinitude. It will try to obey the metrics of both spaces, and the diagram below shows what happens as it moves, the yellow version obeying space and staying the same size and shape in space, while the magenta version obeys the counter space metric.



The counter space- or inner-infinity is shown as a point at the bottom, and lines have been drawn from it through the vertices of the cube. The counter-spatial movement is such that the vertices stay on these lines in order to obey its metric properties, as illustrated by the magenta cube, while the spatial one stays the same spatially. With our ordinary consciousness that is what seems natural, of course, but for a counter space consciousness the other is most natural and the yellow cube appears to be getting bigger (NOT smaller!!). The geometric difference between the two cubes is referred to as *strain*, analogously to the use of that term in engineering where it is the percentage deformation in size when, for example, an elastic band is stretched. The elastic band responds to the strain by exerting a force, which is referred to as *stress*. The central thesis here is thus:

1. **Objects may be linked to both spaces at once,**

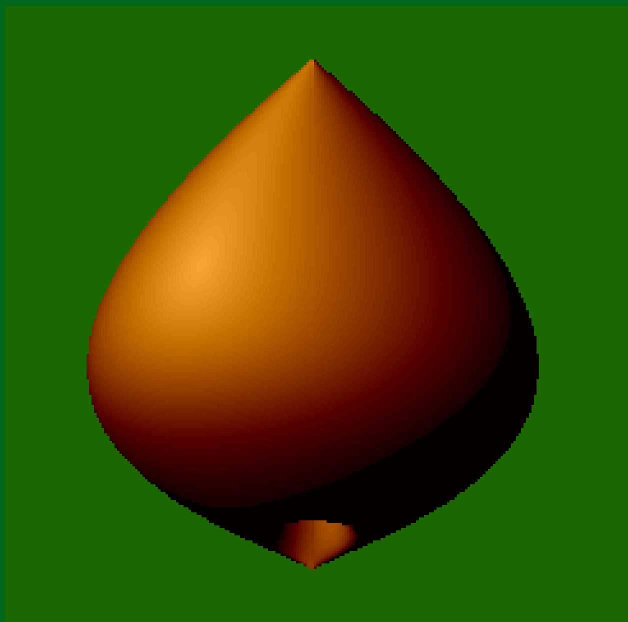
2. **When they are, strain arises when they move as the metrics are conflicting,**

3. **Stress arises as a result of the strain.**

Note well that stress is not a geometric concept, and we move from geometry to physics when we consider stress. The major stress-free movement or transformation is rotation about an axis through the counter space infinity,. which may explain the ubiquitous appearance and importance of rotation in most branches of physics e.g. in fluid flow.

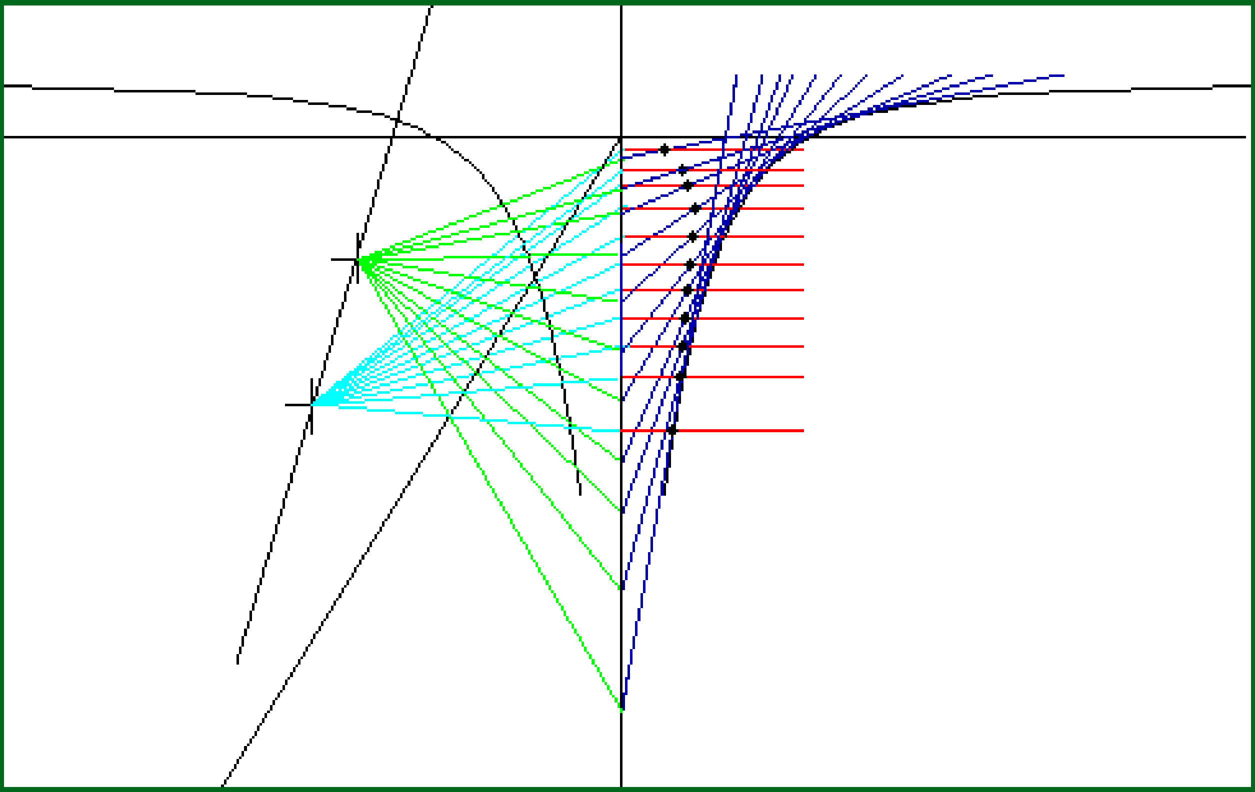
This, and all else in the pages concerned with counter space, is explained in more detail in "Science Between Space and Counterspace" (Reference 11). Some algebraic details are given in the subordinate algebraic page.

Pivot Transforms



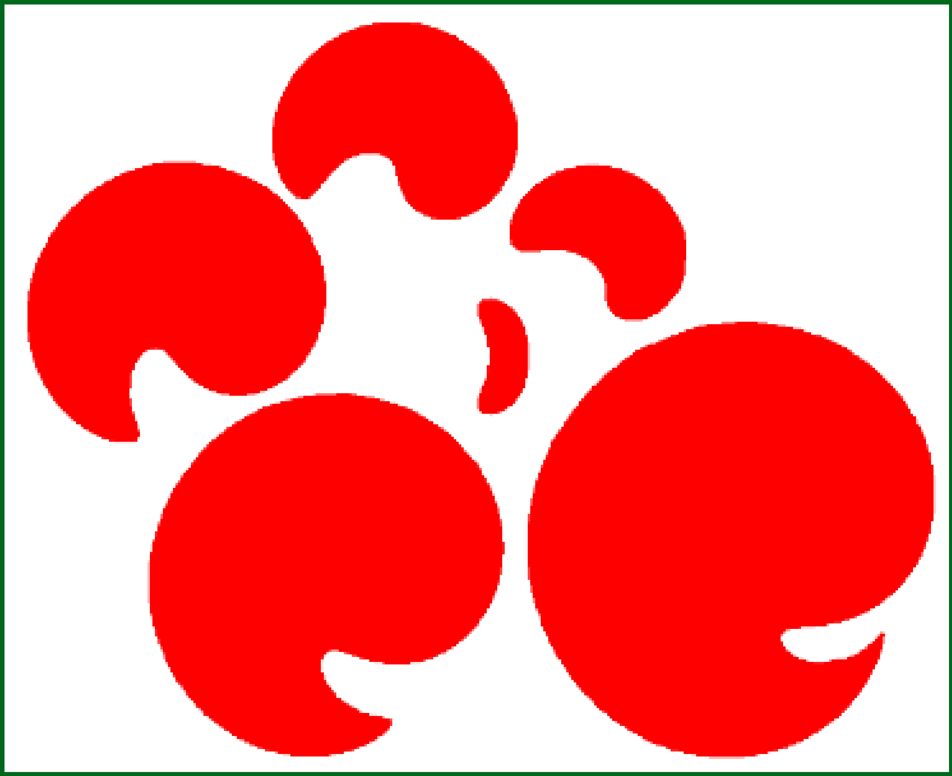
If any plane is placed in a path curve transformation then it is being moved by that transformation. There is generally one point in it which is momentarily stationary, that is, the plane is *pivoting* about that point. It is known as the *pivot point* of the plane. If we place a surface in the transformation then every one of its tangent planes has such a pivot point, and they form another surface known as the *pivot transform* of the first one. They are described by Lawrence Edwards (Reference 7). The author has written a brief summary assuming college level mathematics. The above animation shows how the transform of a vortex varies as its lambda value is varied from -0.9 to -0.1, other parameters being held constant.

Lawrence Edwards discovered these transforms when investigating the shapes of plant seed pods. He found that if a suitably positioned watery vortex is transformed it gives a very good fit. The following animation shows how such a transform may be constructed:



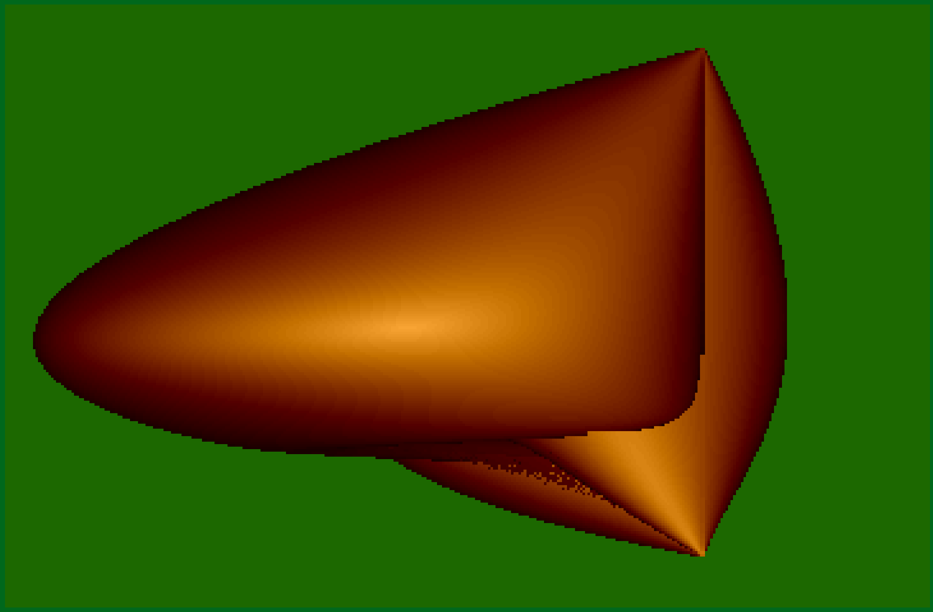
The initial picture shows part of the vortex, the lower invariant plane of the bud transformation as a horizontal line, and two centres of projection and an auxiliary line determined by the bud lambda and epsilon. The final profile is shown by black dots where corresponding blue tangents and red lines meet. The blue lines represent tangent planes orthogonal to the picture, and the red ones horizontal planes.

He then investigated how an airy vortex is transformed and found forms displaying invagination, reminiscent of embryonic forms. He calculated the horizontal profile of the transform of a particular vortex, and as the the vortex axis was rotated the form changed as shown below.



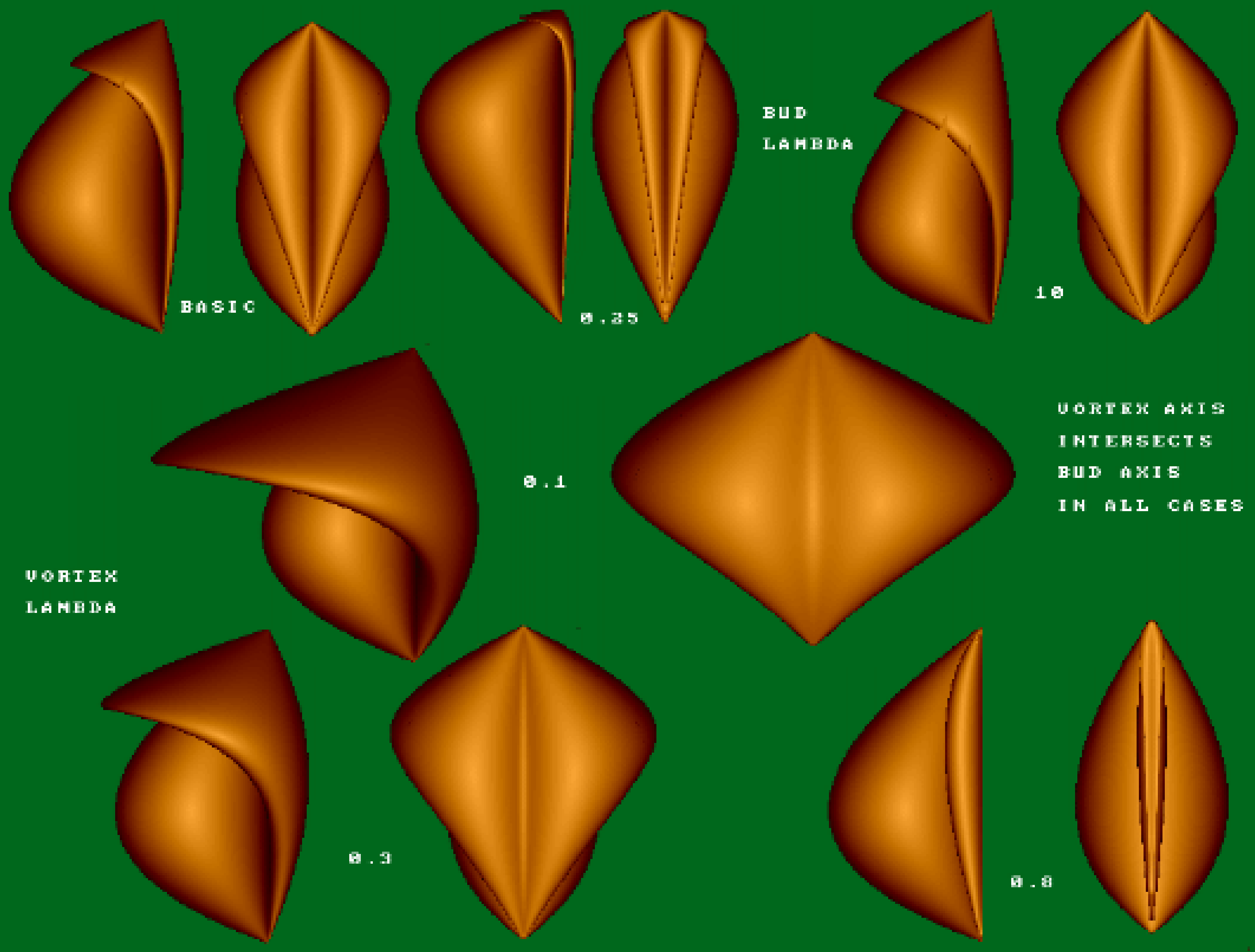
The vortex axis starts at 19° to the vertical at an azimuth of 180°, swinging round to 163° azimuth and 62° to the vertical (for the largest image).

The full three-dimensional forms containing these profiles (which were 30 percent up from the bottom) are shown below:

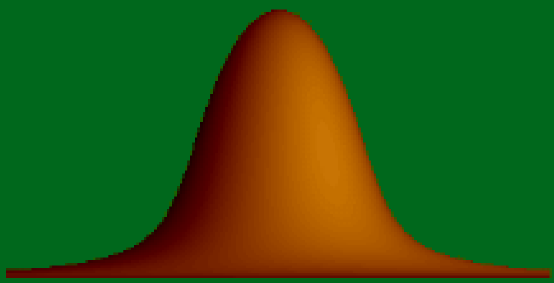
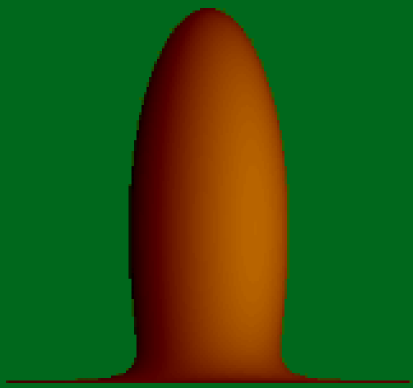


These images were obtained by calculating the angle of the tangent plane at each visible point, and setting the brightness according to its orientation to the direction of illumination. This required a sophisticated bisection algorithm which could not always find the required root of the equation, which is why there are blemishes.

The following image shows some other such forms where the vortex axis always contains the upper invariant point of the bud transformation (hence the symmetry).

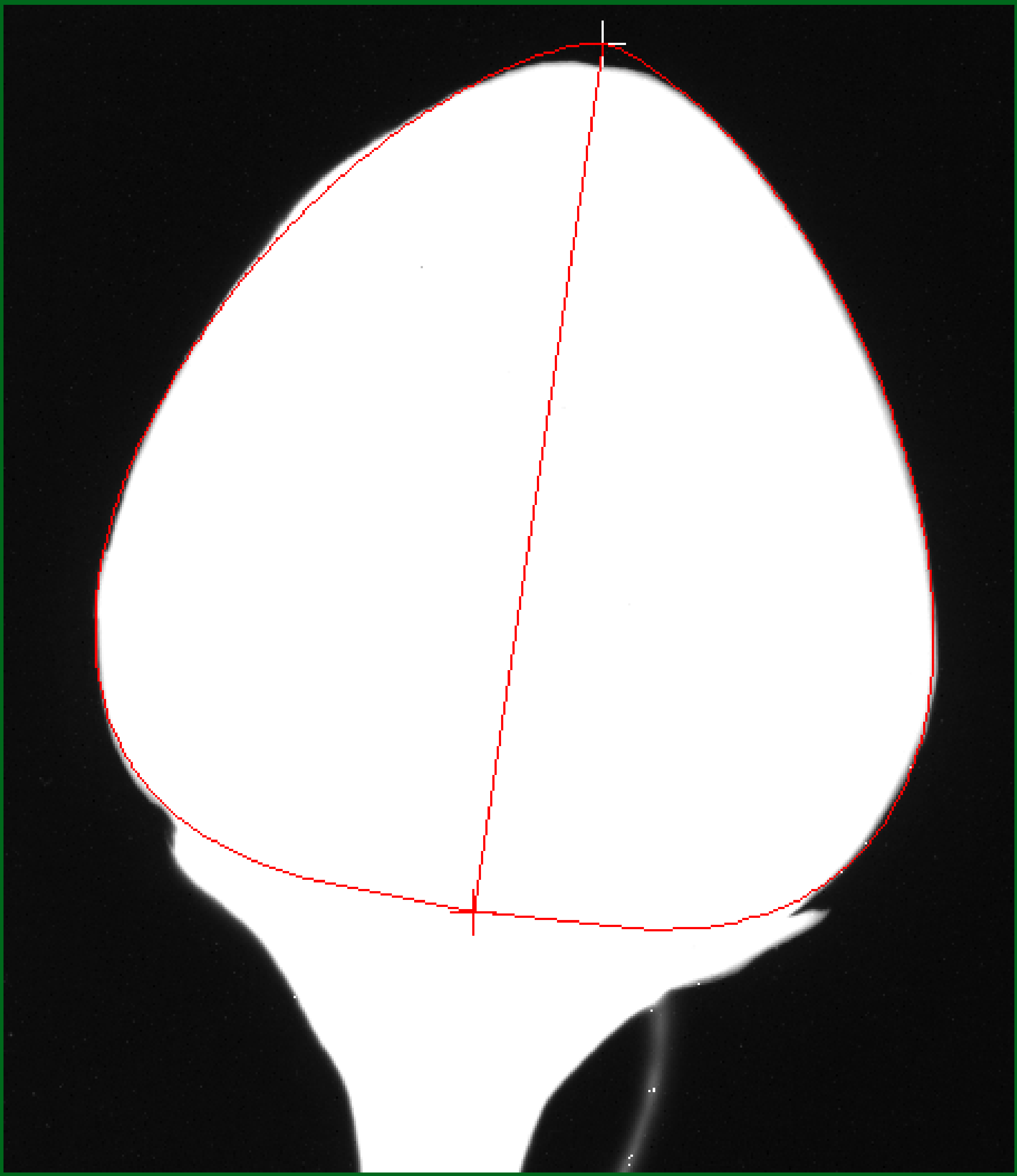


Of course other surfaces can be transformed, and we see for example how bell forms can be obtained from quadric surfaces

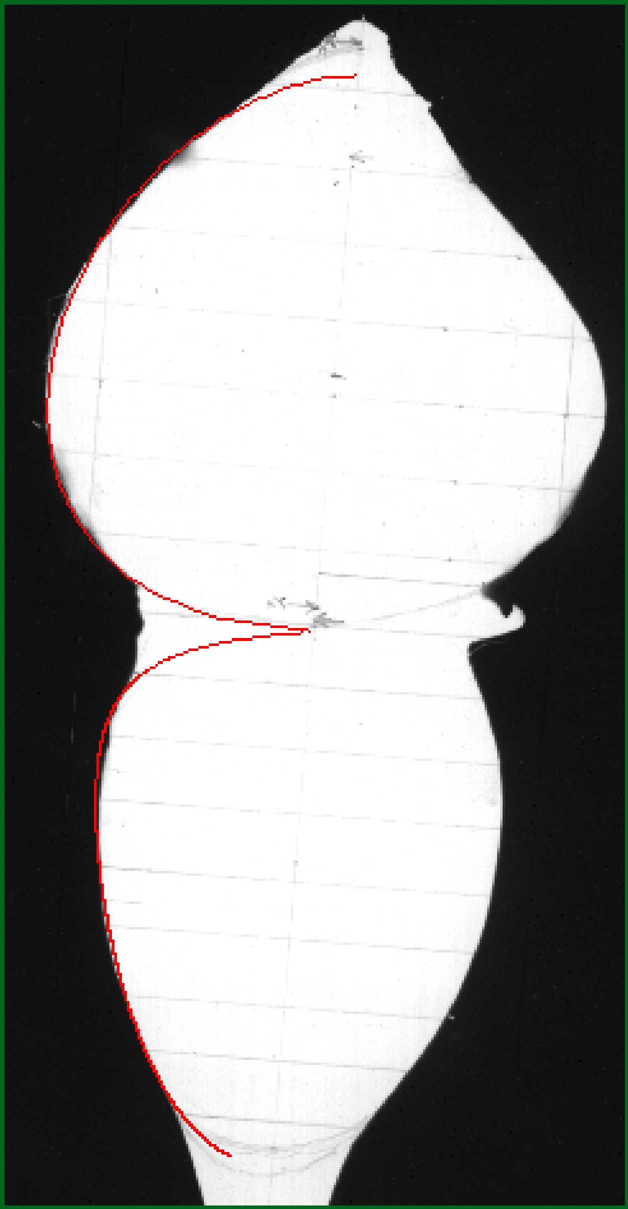
PIVOT TRANSFORMS AS GYNOECIUM FORMS

On the Path Curves page the application of those curves is briefly described. Below are two examples of actual results:



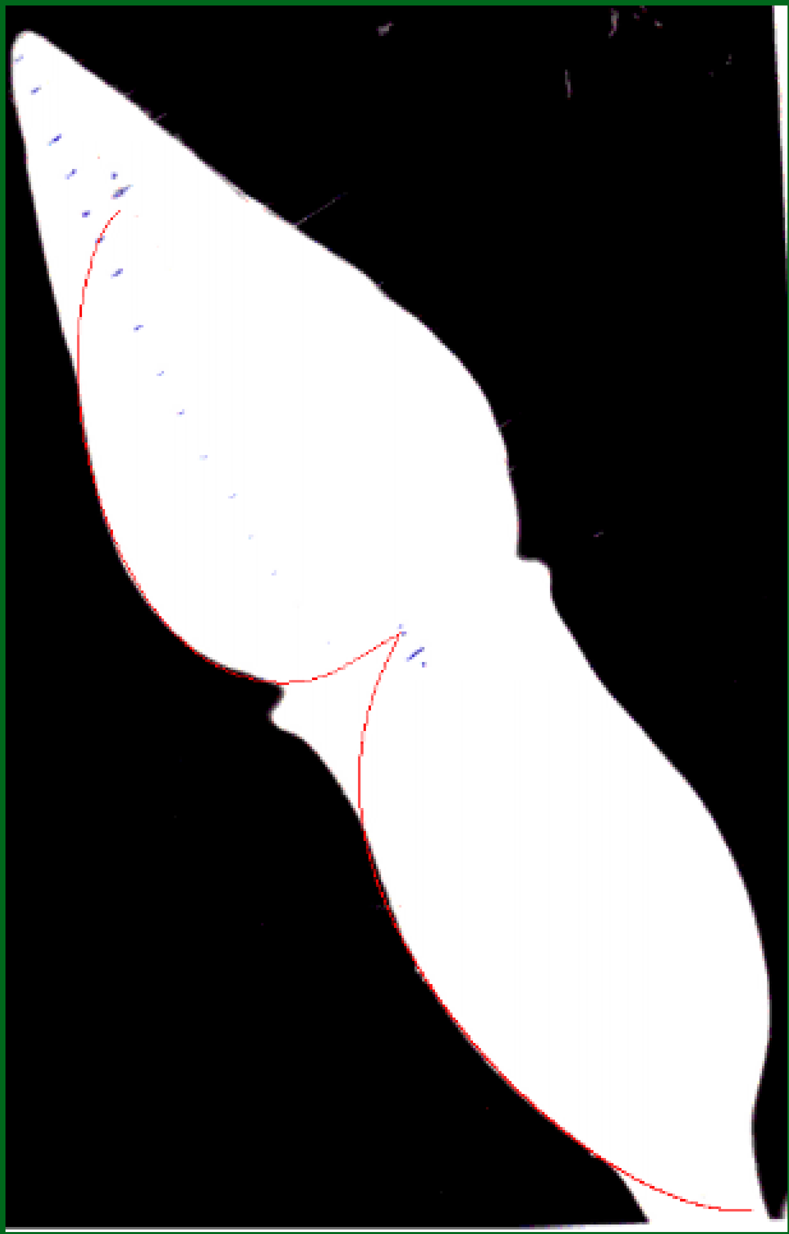
This shows a Kerria Japonica bud with the theoretical curve superimposed in red, which can be seen to be an excellent fit. It is accomplished by selecting the axis by eye together with several points on the profile, and then finding the best mean lambda to fit them. The mathematical curve approaches the top and bottom points (marked by crosses) asymptotically, and we cannot expect an actual physical form to accomplsh that! So the top and bottom points are varied on the axis to minimise the deviation. The top point is in this case above the physical bud, but more usually for other buds there is a physical prominence above the mathematical top. The percentage deviation of the lambda value is 1.2%, a very good result as that is a more sensitive indicator than the mean radius deviation. An added interest in this case was that only the right profile was analysed, yet the resulting fit is also excellent for the left profile. Many buds, like the rose below, are asymmetrical and with a prominence at the top.

Lawrence Edwards discovered the Pivot Transform when seeking a way to describe the gynoecium or seed pod. His idea was to use the projective transformation that produces the bud form to transform another surface. The path curves arise as the invariant curves of a linear transformation, and that very transformation is then used to transform another surface. He found that transforming a water vortex gave the form of the gynoecium (in contrast to the transformation of the airy vortex shown above). The picture below shows a rose bud and its seed pod. As it is asymmetrical the left profile of the bud was analysed, and the resulting fit is shown in red on the bud. Then the transformation corresponding to that was used to find a vortex that transformed into the gynoecium, the result being superimposed on the left side of the seed pod. The closeness of the fit is striking. What is more striking is that this process applies to many buds i.e. in every case it is a watery vortex that is transformed by the bud transformation to give the gynoecium. The vortex is coaxial with the bud, and its invariant plane lies between those of the bud transformation.



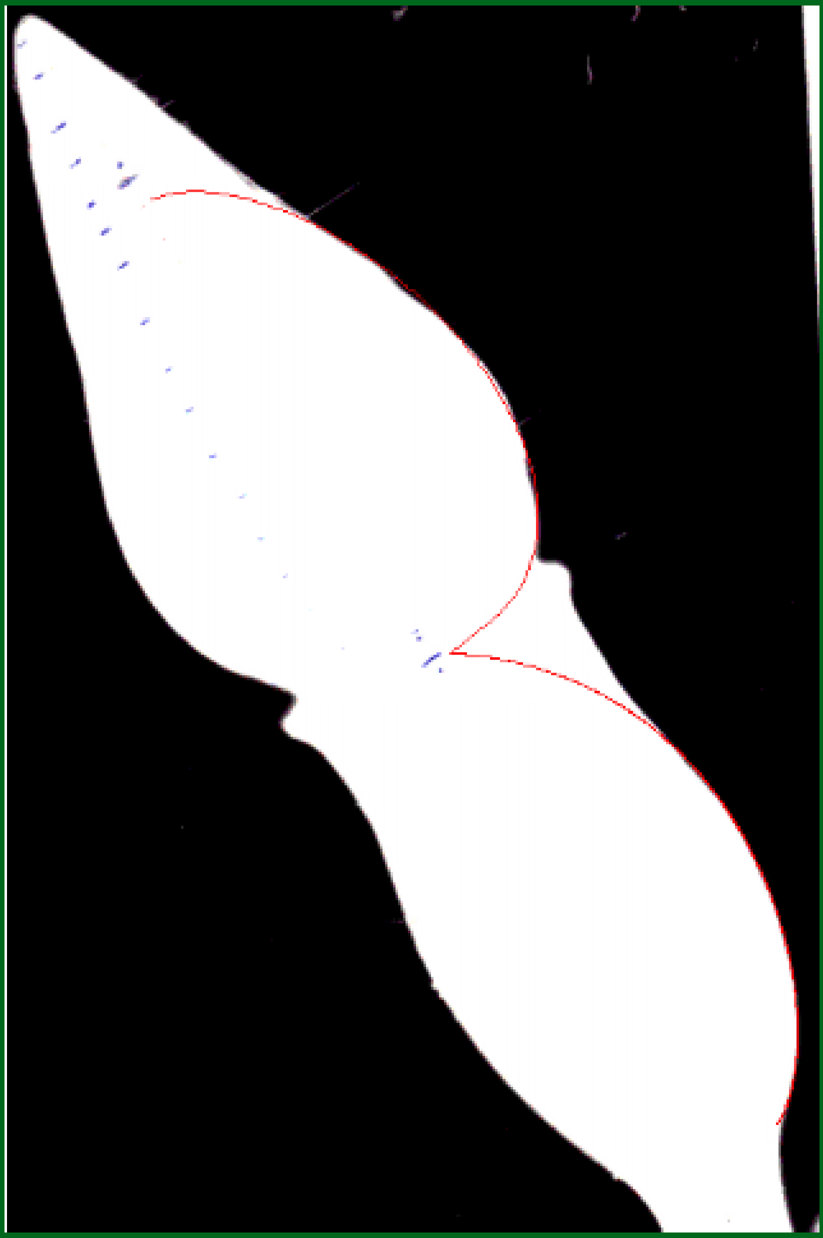
Clearly there is some kind of trade-off between the ideal form represented by the mathematical curves and the physical necessities of actually producing it, together with the required structural integrity which requires a stalk, and a portion between the gynoecium and the bud where the sepals were attached, and so on. The attempt to fit a gynoecium form is very sensitive to the relation between the bud lambda and the actual gynoecium size, and will fail if the lambda is not determined accurately i.e. we do not just get a bad fit, we get none at all as the mathematics fails with imaginary values where we require real ones.

The next picture shows the fit for another rose bud , illustrating that the gynoecium really does depend upon the bud shape and is not just a standard one, as the shape is more elliptic than the above one:



In this case there is a large prominence at the top which evidently is not part of the bud, and any attempt to include it with a bad fit fails to find any gynoecium form at all. It opens up the possibility for such buds of finding a criterion for judging what belongs to the ideal form in physical reality, and reinforces the judgement made by eye, which is easy in this case.

Although the right hand profile is less precise, bearing the above remarks in mind it is nevertheless possible to find a good path curve for it, and a surprisingly good gynoecium fit:



Such results can only excite wonder at the processes occurring in Nature, and how much we have to learn about their holistic aspects which can be investigated with this approach.

The prints shown above were obtained by placing the bud directly in an enlarger to obtain the profile, and the lines drawn on them were for hand calculation of the parameters. However the red curves were obtained by computer methods.

The mathematics of the pivot transform is described in Reference 7 and also in the document Pivot Transforms.

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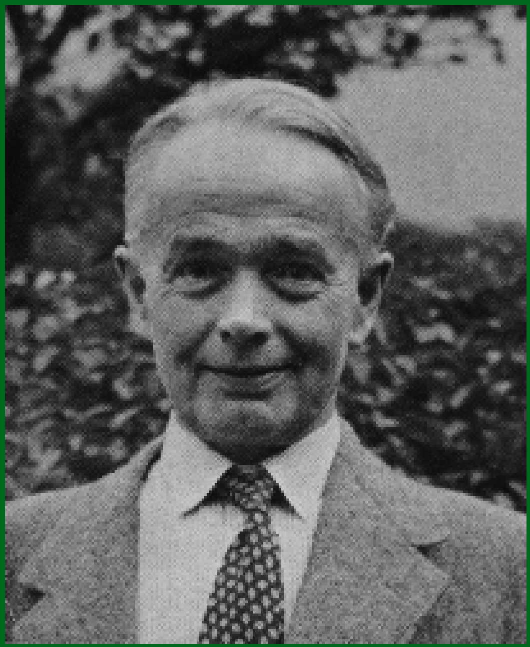
People

RUDOLF STEINER

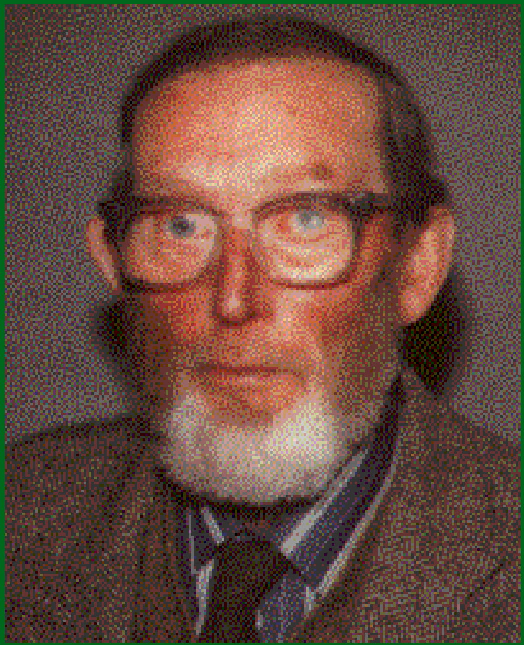
Rudolf Steiner Rudolf Steiner was born in 1861 and lived until 1925. He developed spiritual science by applying the scientific method to his remarkable powers of clairvoyant perception.. When observing subtler aspects of existence he could change his consciousness so that instead of experiencing the world from a central point of view his consciousness moved to the cosmic periphery. He described his findings in over 50 written works and nearly 6,000 lectures. He founded the Anthroposophical Society in 1912 and gave impulses for new more spiritual approaches to agriculture (biodynamic), architecture, the arts, education, care of the handicapped, medicine, science and social science, as well as the path of individual spiritual development. He was born in Kraljevic in Austria (now in Croatia), he read chemistry, natural science and mathematics for his degree and obtained his doctorate in philosophy.

GEORGE ADAMS



George Adams von Kaufmann was born in 1894 and lived until 1963. He read chemistry at Cambridge and came into contact with Steiner's work while a student. He was active as a pacifist in the First World War and did social work with the Quakers, in particular with the Friends' War Relief organisation in Poland. He worked for the rest of his life for Anthroposophy with a special interest in the scientific side as well as developing the social aspects. He interpreted Steiner's lectures in England and later translated many of them into English. He discovered how to describe Steiner's findings about negative space in geometric terms. He worked particularly with projective geometry and the application of path curves.

LAWRENCE EDWARDS



Lawrence Edwards (1912 -2003) studied the work of Rudolf Steiner and as a result he became a Class Teacher as well as an upper school mathematics teacher at the Edinburgh Rudolf Steiner School until he retired. He was inspired to carry out scientific research after studying projective geometry with George Adams, following a "moonlighting" second career testing whether the path curves he had learnt about applied to real forms in Nature. This he confirmed for the forms of many flower and leaf buds as well as for the human heart. He found important rhythmic processes active in leaf bud forms over the winter months which correlate with planetary rhythms. He was a friend, inspirer and helper to many others.

NICK THOMAS

Nick Thomas was born in 1941, educated as an electrical engineer, and became an engineering officer in the RAF for 16 years. He met the work of Rudolf Steiner at the age of 18 and has been inspired by it ever since. In particular he seeks to reconcile Steiner's spiritual research with the findings of science, and has found projective geometry to be a beautiful and appropriate approach. Lawrence Edwards befriended him early on and helped him greatly. Some of his interests and work are outlined in these pages.

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11. *Science Between Space and Counterspace*, N.C. Thomas, Temple Lodge Publishing, first published London January 1999.

CORRECTIONS (downloadable Word 97 document).

A second edition of the book is now available, with the above corrections incorporated.

A new publication is: **Space and Counterspace, A NewScience of Gravity, Time and Light, N.C.Thomas, Floris Books.**

This is a less-technical version of the first book with some added material.

12. "Pivot Transforms", N.C. Thomas (now in PDF format)

and Annex 1 , Annex 2 and Annex 3 thereto (PDF files),

and the diagrams referred to (PDF file).

The main document and Annex 3 have been amended for greater clarity.

13. Practical Path Curve Calculations, N.C. Thomas (in PDF format).

14. *Algebraic Projective Geometry*, Semple and Kneebone, Oxford University Press, Oxford 1952.

15. *Projective Geometry*, T.E. Faulkner, Oliver and Boyd, Edinburgh and London 1960.

Selected Other Sites

Healing Water Institute

Bud Workshop by Graham Calderwood

Projective Geometry and Life Forms

Goetheanum

- Mathematical-Astronomical Section (In German)
- Natural Science Section (In German)

Astronomy Picture of the Day

Ifgene

Canadian Mathematics Society

What is Science?

Rudolf Steiner Press

Temple Lodge Publishing

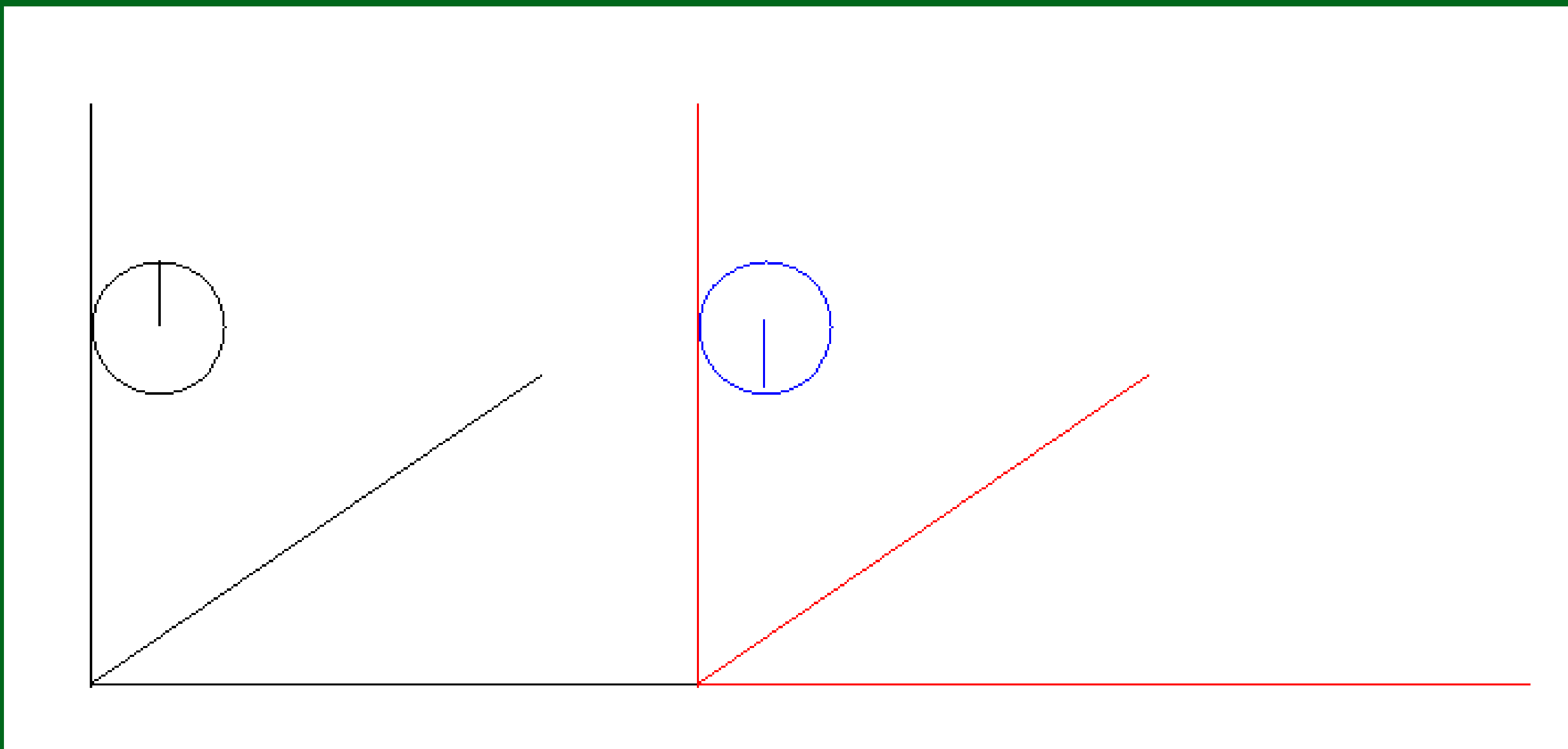
Freeware Network for free software

- Desk Multi-client GUI widget server for plain C programs

Recent Changes (Apr 2009)
This page: FTP documents replaced by PDF files

This page may be off-subject and generally volatile i.e. for temporary items.

ARE TIME MACHINES POSSIBLE?



Time machines are topical, with articles in popular magazines suggesting that the Large Hadron Collider (LHC), due to start operation later this year, may produce wormholes enabling time travellers from the future to reach back to the moment when that first happens. Well known paradoxes are raised by the possibility of physical time travel to the past, such as a man murdering his grandmother and so on. Fascinating science fiction stories have been written about the subject, beginning of course with H.G.Wells' *The Time Machine*. A nice twist is when the inventer of a time machine travels back to the moment when it is invented, and publishes the patent. Some physicists eagerly accept the possibility of time travel while others, such as Stephen Hawking, do not. So does physical time travel make sense? Two reasons will be suggested here why not:

> A misunderstanding of time-dilation in Special Relatvity that goes back to Einstein himself;
>
> Time is assumed to be a dimension, which is not necessarily true.

Time Dilation

In his Special Theory of Relativity, Einstein sought to meet two objectives:

> that physical laws are the same in all inertial reference systems;
> that the velocity of light in a vacuum is constant regardless of the state of motion of an observer.

The first means that there is no absolute frame of reference for which physical laws are simplest, but rather they are the same in all reference systems that are in uniform rectilinear motion with respect to each other. The second means that the velocity of light will appear to be the same in all such reference systems i.e. the observer's velocity is not added to that of light. Three startling consequences of the equations of motion that solved this programme are:

> Moving objects increase in mass, which becomes infinite at the speed of light;
> Moving objects become shorter in their direction of motion, shrinking to zero at the speed of light;
> Clocks on a moving object appear to tick more slowly as seen by outside observers, stopping altogether at the speed of light.

The third consequence is called *time-dilation*, and it should be appreciated that it does not only apply to clocks. All cyclic or rhythmic processes will appear to slow down, including the beating of a human heart. Einstein concluded that time itself slows down for the moving object relative to outside observers. The famous twins paradox is based on this, where one twin (Fred) stays at home and the other (Jim) accelerates to a speed near that of light, travels for several years, reverses velocity and travels back home again. Because Jim's heart appears to slow down he appears to be younger than Fred when they are reunited. The paradox lies in the fact that the same argument can be applied to Fred as seen by Jim, so that Jim expects Fred to be younger. However there is a flaw, as while Fred may well be in an inertial frame of reference, Jim most certainly is not because of the accelerations he undergoes, and General Relativity may be invoked to show that Jim will in fact be younger than Fred because of that.

Einstein's (or Lorentz's) equations do not say that time itself slows down, only that time intervals will appear to be longer, for Einstein banished the notion of absolute time, so time as such is not involved, only intervals between events. An experimental confirmation of this idea is that particles called *muons* arising from cosmic rays entering the atmosphere reach the surface of the

Earth in greater numbers than expected. That is because they decay quickly, having a definite half-life, which enables the expected number of arrivals to be calculated. The observed rate of arrivals suggests that the muon's "clocks" are ticking about 9 times more slowly as observed on Earth than those observed in the laboratory, and so they live long enough to reach the Earth. Now the half-life of their decay is based on internal physical processes, which time-dilation shows should slow down thus increasing the observed half-life. Now this is a purely physical statement about process rates, and need not imply time itself goes more slowly for the muons. Denying that time itself is affected does not invalidate any of the experimental findings supporting Relativity, but it does suggest that no "time travel into the future" is involved. This is more fully explained in an article which may be downloaded (see below).

Time as a Dimension

Einstein also treated time as a fourth dimension alongside the three of space (ignoring for now the extra dimensions assumed by Superstring Theory). However it is argued in the accompanying article, which may be downloaded, that this leads either to a static universe with no genuine evolution or change, or else to an infinite regress as an extra time-like dimension would be required to measure changes occuring in four-dimensional space-time, etc, etc.. This dilemma is solved if time is not assumed to be an extra dimension alongside the three of space.

Physical observations can never distinguish between those two views.

Wormholes are supposed to be topological distortions of space-time predicted by General Relativity that enable "short cuts" to be taken as in science fiction. However if time is not a dimension then the dramatic changes in the rates at which physical processes proceed at either end of a wormhole are just that, but imply no time travel in the sense of H.G. Wells. Similarly the drastic changes in process-rates supposed to occur in the vicinity of a black hole are again just that - changes in process rates - without time itself being affected.

SO: if time itself does not slow down for moving observers, but only rates-of change of processes do so, and time is not itself a dimension, then the physical possibility of time travel does not arise.

ARCHIVE

TETRAHEDRAL COMPLEX

OTHER REPRESENTATIONS OF GEOMETRY

COMET IMPACT

DOUBLE LINES

Simple Chaos Theory

Asymptotic Lines

Covariance and Contravariance

The article on pivot transforms applied to gynoeciums has been relocated on the PIVOT TRANSFORM page.

Back Home

Contra Time Machines

N C Thomas, C Eng, MIET

Introduction

The possibility that time machines could be constructed is taken seriously by the physics community, although the resultant paradoxes cause unease to many. This depends critically upon two assumptions:

> that time is a dimension alongside the three familiar spatial dimensions;
>
> that time dilation predicted by Special Relativity and verified experimentally implies time itself is affected by relative velocity;

In this article both these assumptions will be challenged without thereby invalidating what is physically essential in the Special and General Theories of Relativity. The result is that time machines are not possible in the sense usually envisaged.

Special Relativity

Albert Einstein developed his Special Theory of Relativity to meet two requirements or postulates:

> Physical laws should be invariant with respect to uniform rectilinear motion
>
> The velocity of light is constant *in vacuo* for all observers regardless of their state of motion

The first says that the laws of physics should not be affected by uniform relative motion, so that the behaviour of a pendulum, for example, should be governed by the same physical factors and the same mathematical equation relating them in all inertial systems i.e. systems in uniform rectilinear motion. Other examples are that we do not expect the law of conservation of energy to be correct in only one reference system, we do not expect fluids to become gases just because their containers are in relative motion, and so on. In short there is no absolute reference system for which the laws take their simplest form: they have that form in all inertial reference systems. Another way of saying this is that we do not expect Nature to be affected by the way we describe her (mathematically). Einstein himself said that he would find it "distasteful" were it otherwise (Reference 1).

The second requirement was adopted for a number of reasons, some theoretical and at least one experimental. In the 19th Century it was supposed that light waves must have a "bearer" medium analogous to the fact that waves in water, for example, must have water to bear them. This bearer or medium was called the *ether*, which was supposed to pervade all space and to have suspiciously ideal physical properties. Michaelson and Morley carried out a famous experiment in the 19th Century to detect the movement of the Earth through the ether, but obtained a null result: no such movement was detected. While there may be a number of interpretations of this remarkable result, the concensus from Einstein onwards is that there is no ether and that the velocity of light *in vacuo* is the same regardless of the state of motion of an inertial observer. Its velocity <u>is</u> supposed to be reduced when travelling through a medium such as air or glass, and the phenomenon of refraction is explained on that basis.

Einstein developed a set of equations governing the relative movement of inertial systems which satisfy the two postulates. They are essentially rooted in *tensors*, which are special mathemematical entities that permit laws to be expressed in a form that does not depend upon the coordinates of space and time used. For example we may select as our coordinate system the position of an object relative to London so that one measurement is along a line (an axis) running north/south through London, another axis is east/west and the third vertically up and down. Together with time we then have a coordinate system. Or we may choose to centre our system in the Sun, at the centre of gravity of the Solar System, with one axis through the vernal equinox, one at right angles to that in the Ecliptic, and the third at right angles to the Ecliptic. Again, together with time we have an equally valid coordinate system. Should Nature make her laws depend upon which of these two systems (or any other) that we select? Einstein thought not, which is the basis of the first postulate above, and tensors are a terse and elegant way of expressing that fact.

As an aside, a problem with London is that the Earth is rotating, so strictly speaking such a coordinate system is not inertial, but as it is very hard to find a familiar example we let that example illustrate broadly what is involved. The Sun based system is not exactly inertial either as the Solar System is moving round the centre of our galaxy rather than on a straight line. So the concept of an inertial system is abstract and it is hard to find one in practice. When General Relativity is taken into account this problem is actually eased because it handles acceleration as well as uniform rectilinear motion.

The tensor equations may be cast into a more transparent form, and for example that governing how velocities should be added is

$$v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$$

where v_1 and v_2 are the velocities of two objects relative to an observer, c is the velocity of light and v is the relative velocity of the two objects as measured by the observer. Thus if $v_1 = c$ or $v_2 = c$ (or both) than $v = c$, showing how the second postulate is satisfied. Of course the two objects are travelling along the same straight line in this example, but it is readily adapted for other cases. c is an upper limit which cannot be exceeded, or even reached by massive objects.

Now suppose an observer A has a relative velocity v with respect to another observer B, and each observes an event E. Einstein showed (Reference 1 for an accessible account) that if E occurs at a distance x from A at a time t, then the distance x' of E from B in its own coordinate system is given by

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where t is the time since A and B coincided. The time t' of the event for B is given by

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}}$$

which contrasts strongly with our intuition that t=t'. These two equations were actually first derived by Hendrik Antoon Lorentz in 1904, but Einstein gave a convincing rationale for them.

Suppose now that the event is the moment when a pendulum at rest with respect to A is at the bottom of its swing, and that it has a peroid T as seen by A, so that for A two successive such events occur at times *t* and *t+T*, the time difference trivially being calculated as (*t+T*)-*t*=*T*. For B the time difference is

$$T' = \frac{t + T - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{T}{\sqrt{1 - \frac{v^2}{c^2}}}$$

i.e. $T' > T$ so that the pendulum appears to be swinging more slowly for B. If that pendulum is part of a clock then the clock appears to tick more slowly. The above calculation applies to all cyclic or rhythmic processes, including any type of clock, biological processes and so on. Thus a person's heart will appear to beat more slowly too, and if $v=c$ it will appear to stop altogether as T' becomes infinite. This is the basis of the famous twins paradox, where one twin remains on Earth and the other travels away at a velocity close to *c*, and thus appears to age much more slowly. The catch is that if the travelling twin reverses his velocity <u>the same happens on the return journey</u>, for then we have

$$T' = \frac{t + T + \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{t + \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{T}{\sqrt{1 - \frac{v^2}{c^2}}}$$

i.e. as before clocks also tick more slowly. This contrasts with the Doppler shift, illustrating that the two phenomena are quite distinct. Thus on his return it seems that the much-travelled twin will be younger than the stay-at-home one. The paradox here is that the travelled twin should observe the other aging more slowly in an analogous way, so that in the end they should not differ in age. This has been the subject of much heated debate! See e.g. Reference 2. However sticking to Special Relativity is insufficient to describe the whole affair, as the travelled twin is not in an inertial reference frame!!! Why this simple fact is so often ignored is a mystery to the author. For accelerations are involved to set off, reverse velocity, and slow down at the end. General Relativity is required to account for the effects of acceleration (the paradox is resolved on this basis in Reference 3).

Time

We will now take issue with a conclusion Einstein drew from this which we claim need not be true. He said that because clocks tick more slowly <u>therefore time itself slows down</u>. However his equations do not show that, what they do show is that cyclic and rhythmic processes are slowed down, which has been thoroughly verified e.g. by the extended apparent lifetimes of muons in the atmosphere. It no more follows as a logical necessity that time is slowed down than that it would be if we simply lengthened the pendulum of a clock to make it tick more slowly. Clocks do not determine time! That would be the proverbial tail wagging the dog. This may seem like a merely philosophical point, but whatever label is attached to it, it is a very important point. It is intimately connected with the notion that time is a dimension which we travel through, somewhat analogously to the fact that we may travel through a spatial dimension. That the variable *t* enters into all equations involving motion appears to justify that notion, and physicists speak of space-time as a four-dimensional continuum. Einstein wrote that there is no essential difference between these four dimensions and that time only seems different to our kind of consciousness.

Returning to the experimental confirmation of time dilation provided by the extended life of μ mesons, these particles have a half life of 1.53×10^{-6} seconds. Experiments indicated that more muons arising from cosmic rays entering the atmosphere arrived at the surface of the Earth than would be expected based on that half life. In fact the half life was about 9 times its laboratory value due to time dilation (Reference 3). Now the half life results from processes in the muon that lead it to decay, and if those processes are slowed down then it will have a longer half life. We do not have to conclude that time itself slows down in the muon rest frame.

But is time really a dimension? Is there any evidence for that? What we know is that we require a variable *t* in order to calculate velocity and other variable quantities, and that it seems that *t* is steadily increasing. It does not really matter whether it increases steadily or not as <u>it</u> is the yardstick for change. If it increases in some other way, how would we know? By means of another dimension? This requires us to move on to General Relativity.

General Relativity

Obviously it would take too long to give a full account of this here (see Reference 4 for a useful account), but something is needed if we are to discuss time travel. Einstein pointed out that there is no physical experiment that could exhibit the fact that one is undergoing a rectilinear translation, and in his example a person in a closed box travelling uniformly along could not determine that fact. For example a pendulum would not reveal it for the reasons we have already seen: its laws are the same in all inertial reference systems. However if you are sitting in an aircraft and it starts doing aerobatics you certainly know who is accelerating, and even if you are a bit slow on the uptake, your stomach is not! This is why Special Relativity only considers inertial reference systems. Now the first postulate of Special Relativity was

> Physical laws should be invariant with respect to uniform rectilinear motion

Surely we should not stop there! We would like to say something like

> Physical laws should be invariant in all reference systems

But then we must somehow explain the situation in the above aircraft. This is exactly what Einstein did, as follows. Returning to his closed box, suppose you are in such a box and unknown to you it is being pulled along in empty space by a rope (we'd better not ask just how!) such that it is undergoing a uniform acceleration of 1g. If again you observe the motion of a pendulum in the box it will behave exactly as it would on the surface of the Earth i.e. you could not tell whether you were being accelerated or were in a gravitational field. Einstein then postulated that <u>there is no difference</u>. In other words gravity and acceleration are equivalent in all respects. The mathematics required to capture this idea is beautiful and difficult, based again on tensors for the reasons explained before. What emerges is that space-time is curved both by gravity and by acceleration. This curvature is exhibited by the fact that light does not travel on a straight line in the presence either of a gravitational field or an acceleration. In the accelerated closed box a photon travelling across the box on a path starting e.g. at right angles to the direction of motion will follow a curved path. The idea was verified by Sir Arthur Eddington and others in 1918 during an eclipse of the Sun, when stars seen close to the perimeter were displaced outwards compared with their normal positions. Also an anomalous precession of the perihelion of the planet Mercury, previously unaccounted for, could be explained by the equations of motion given by General Relativity. Furthemore the elliptic orbits of the planets round the Sun could be shown to arise from the curvature of space caused by the intense gravitational field of the Sun. Many tests of both Special and General Relativity have corroborated them. (Indeed no falsifying experiment is known to the author, although the entanglement of photons in Alain Aspect's experiment in 1982 to test Bell's inequality seems to approach that. It is said that no signal could be transmitted faster than light by that means, the point being that permanent observation of the polarisation awaiting its determination at the other end is not possible, and otherwise it is not possible to know when to test that it has been determined by an observation at the other end without receiving some other signal to say so. But it remains true that the determination of the polarisation has been transmitted faster than light, even if that is not practically usable).

So far so good. What about an object falling vertically downwards towards the Earth, does it not follow a straight line? For the point now to be made, we ignore the movement of the Earth round the Sun, and of the Solar System in our galaxy, which would suggest otherwise. The line is then straight relative to the Earth, and certainly would be were the Earth alone in the universe. So where is the curvature? It lies in the fact that the object is accelerating, following a curved *world line*, which is a special line in space-time. *Geodesics* are, in a curved space, the equivalent of straight lines in a flat space. On the surface of the Earth, for example, the geodesics are (ideally) great circles. General Relativity says that objects follow geodesics in space-time, which replaces Newton's First Law that an object remains in a state of rest or of uniform motion unless acted upon by an impressed force. There are, in curved space-time, so-called *null geodesics* which have zero length. There are no such geodesics in a flat three-dimensional space, but when time is included as if it were a dimension then there are such entities, and they are of great importance. For light travels along null geodesics. This is what distinguishes light (and other radiation) from massive objects.

Thus every object is moving on a world line which is a geodesic, the planets on their (roughly) elliptic geodesics being examples. More accurately, an object *is* a world line, for the movement is only apparent according to this view, since it only arises when one dimension is taken as the reference for change in the others, that dimension being time. We cannot speak of a space-time movement without invoking some other reference, such as yet another time-like dimension. For movement, indeed any kind of change, requires a time-like reference. The conclusion is that the universe is a static assemblage of world lines: change is only apparent as an artefact of our consciousness (according to Einstein). If the universe is to evolve, expanding as is supposed, and is not static then the world lines must be developing and changing as it evolves. But then, as we have seen, we need another reference dimension. So we end up with an infinite regress, for then we will have a five dimensional universe which is in its way static, with more complex world lines, or else undergoing change requiring a sixth reference dimension, and so on. We are left with two broad alternatives: if time is a dimension then we must accept a static universe, or else we must renounce the assumption that time is a dimension in order to evade the infinite regress.

The fact that Relativity has not so far been satisfactority combined with quantum physics, even by superstring theory, and that the mathematical concept of chaos has become respectable and unavoidable, suggests that the static universe view is incorrect. If we accept that conclusion then we must renounce the assumption that time is a dimension. This does not invalidate the experimental evidence in favour of Special and General Relativity, for it is clear that processes do slow down for moving objects and that light does follow curved paths in gravitational fields. The variable t is needed and is part of the mathematical descriptions given by physics. But we now claim that t is not a measurement of a coordinate in a dimension. Whatever time is, it is not a dimension, but it is a reference entity.

It is instructive to review the use made of the Lorentz equations to calculate T' for a pendulum. The essential conclusions of Relativity depend upon differences, where we had the difference between $(t+T)$ and t, and similarly for T'. Einstein insisted that he had swept away the notion of absolute time, so that t should never enter our equations in that guise. We used t as the time elapsed since the two observer reference frames A and B were coincident, so that it too was really also a difference in that case. This is why cyclic and rhythmic processes are readily described and understood *vis-a-vis* time dilation. But if t never arises other than in time differences, then it does not truly play the role of a coordinate. Time, it seems, measures process rates rather than coordinate positions.

Thus if we consider an object approaching the event horizon of a black hole, for example, we can well say that on-board processes will appear to slow down for outside observers, without the need to add that time slows down.

Physical observations can never distinguish between those two views.

In which case there is no empirical case for saying that time itself “slows down”. We lose nothing essential from our conclusion, other than time machines.

The Light Cone

Since no physical effect is supposed to be propagated faster than the speed of light, a three dimensional hypercone in four-dimensional space-time is envisaged with its vertex at an observer such that its surface demarcates objects causally connected with that observer from those that are not. Objects outside the cone are separated from the observer by space-like intervals, those inside it by time-like intervals. Light is taken to be the demarcator between these types of interval as it travels along null-geodesics in the surface of the cone. It is then assumed by some (e.g. Reference 5) that if we could travel faster than light we would violate the demarcation of the light cone and travel backwards in time, as though overtaking light affects time itself. We would then see the past when light catches up with us as if we were actually there i.e. back in time. This forgets that the light has left the scene of the events that could thus be viewed. It is discussed because light has been slowed down in the laboratory almost to a stop, suggesting theoretically that something could then overtake it. It also relies on the assumption that velocity affects time itself, which we deny.

Wormholes and Time Machines

It has been claimed (e.g. Reference 5) that somebody travelling at near light speed “time travels into the future” due to Einstein's equations. Well, if one were willing to concede that somebody in cryogenic sleep for 100 years has time-travelled when unthawed, that loose statement could be accepted. But that interpretation is emphatically not the kind of time travel envisaged e.g. by H.G. Wells in his novel *The Time Machine*. There it is supposed that one may travel through time analogously to travelling along a space dimension. The distinction is very important if the physical possibility of time travel is to be assessed. Our interpretation of time dilation as the slowing down of physical processes rather than a slowing down of time itself leaves the conclusions drawn from Einstein's equations unaffected, as we have already said. But it denies that time travel has thereby occurred in the sense of H.G. Wells. We will now turn to claims that time travel into the past is physically possible, with all the paradoxes and problems that would then arise.

Since space-time may be curved, it follows from General Relativity that it is possible to alter the topology of space-time to include "tunnels" across space-time, like short-cuts from one world location to another. These tunnels are called wormholes, linking two locations in a non-causal manner, and it is supposed that time travel into the past could be accomplished by means of them. This is because time is assumed to be a dimension so that the radical time dilations involved would produce time travel. The catch is that enormous energy is required to create them, as may appreciated by noting that the large mass of the Sun only caused a deviation in the path of light by a fraction of a second of arc in Eddington's observations in 1918. The energy required to roll up a worm hole is thus seriously huge and should give rise to an enormous inertia (as indeed superstrings should have an enormous inertia).

If time is not in fact a dimension then Wells-style time travel is not in question, and the concept of wormholes needs re-interpreting. This demands that we re-interpret the meaning of space-time curvature. What would be observed physically is that light (and other radiation) and physical objects would follow curved paths in space accompanied by alteration in the expected frequencies of processes (clocks etc.). This is what the equations actually predict physically. But processes may alter their rates without that implying time itself is going faster or slower, as we have already noted. It is just time that is the yardstick for rates of change, not *vice-versa*. Then a wormhole would radically deviate the paths travelled by radiation and objects in its vicinity, and also the frequencies of processes. The fact that processes at one end are much slower than at the other does not constitute time travel if we adopt the above physical interpretation of the equations. I cannot murder my grandfather by making my clock go backwards very rapidly, for I remain causally disconnected from him. Likewise a wormhole, if such existed, would not have acausal implications just because it radically altered the rates of change of physical processes at one or both ends.

Many Worlds Hypothesis

The many-worlds hypothesis is not considered as a solution of the time-travel problem because it violates the conservation of energy i.e. if parallel universes arise as the result of a quantum interaction where does the vast amount of energy come from to create them? This problem is severe considering the large number of interactions continuously occurring across the whole universe.

Conclusion

In conclusion we are saying that physical processes obey Einstein's equations without the implication that time itself is affected, and that time is not a dimension. Genuine Wells-style time travel is not in question.

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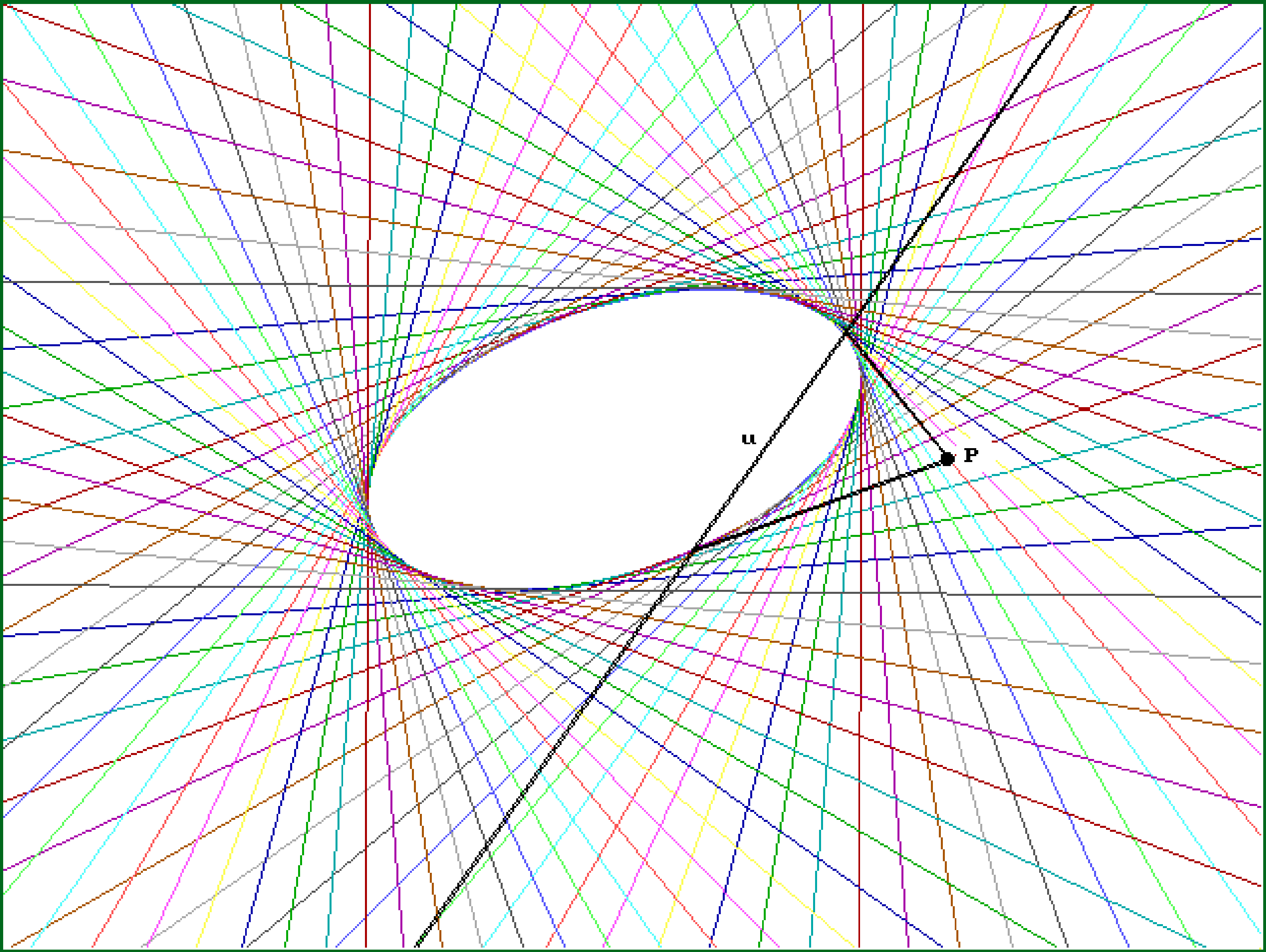
TETRAHEDRAL COMPLEXES

Another branch of projective geometry concerns lines. There is a four-fold infinity of lines in space, of which we may form a subset. A subset containing a threefold infinity of lines is called a LINE COMPLEX. An example which is simple to define is the TETRAHEDRAL COMPLEX: given a tetrahedron, a general line in space cuts its four faces in four points:

TETCOMP3.GIF

These four points have a cross ratio which may be any real number. We may select the set of lines all of which intersect the tetrahedron in the same cross ratio. Since there are infinitely many possible cross ratios we thus select a three-fold infinity of lines from the four-fold infinity of all possible lines. The resulting line complex has a definite structure such that through any point of space it possesses a set of lines forming a cone, while in any plane of space it possesses a set of lines enveloping a conic.

Just as we have polarity wrt (with respect to) conics and quadrics, so we may have polarity wrt a line complex. This means that if we choose any line u then the complex determines a line u' polar to u. This is accomplished by taking the axial pencil of planes in u, and for each such plane finding the point P polar to u wrt the conic of the complex in that plane:



The points P in all the planes of the pencil lie on a straight line u' which is the polar of u. (If u happens to be a line of the complex then it is self-polar).

We may then find the polar of u', which is a third line u'', and so on. An interesting question then arises: what figure is formed by such a sequence of polar lines?
The answer turns out to be quite simple: it is a ruled quadric which is self-polar wrt the tetrahedron. This means self-polar in the sense that the faces of the tetrahedron and their opposite vertices are harmonic wrt the quadric. Although we started with a discrete set of lines u,u',u''... it turns out that if we take any line v on a self-polar quadric Q then its polar line v' wrt the complex also lies on Q.

Since we could have chosen any cross ratio to define the complex, and since a quadric Q is self-polar wrt the tetrahedron irrespective of that cross ratio, we see that the lines on Q form a self-polar set for all possible tetrahedral complexes sharing the same base tetrahedron (such complexes are known as COSINGULAR COMPLEXES). Of course a given line v of Q will have different lines of Q as its polar for different cosingular complexes.

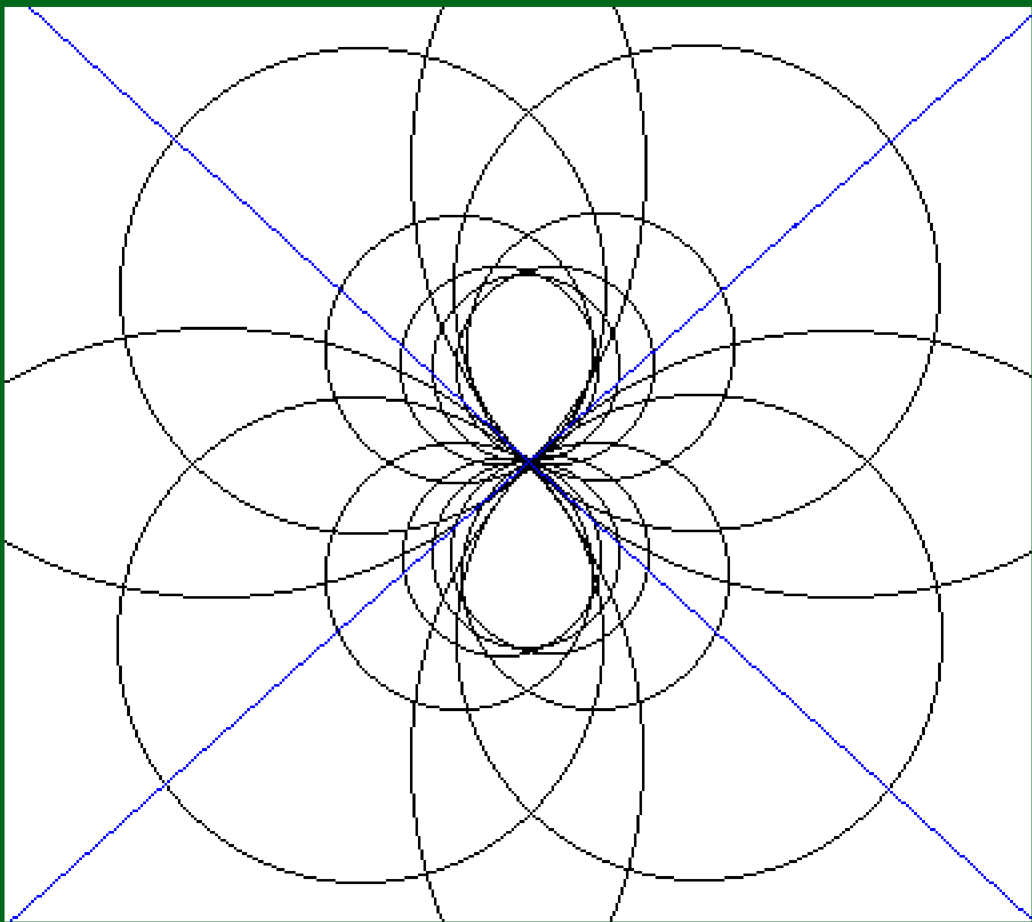
I found this result myself and have not seen it anywhere in the literature. Has anyone seen it published elsewhere?

The proof is available from me (via email).

<<BACK

OTHER REPRESENTATIONS OF GEOMETRY

Projective geometry does not have to have points and lines as its basic elements. For example circles through a fixed base point Z, and points, may be used instead. Just as any two lines meet in one point, so any two circles through Z meet in just one other point. Dually just as any two points determine one line so any two points together with Z determine just one circle. We may then expect analogues of the basic theorems of projective geometry to apply to such a geometry of circles and points. The following diagram shows the construction of a "conic" in this geometry, where two projective ranges give rise to a set of "lines" (circles) joining them, enveloping a "conic" (lemniscate).

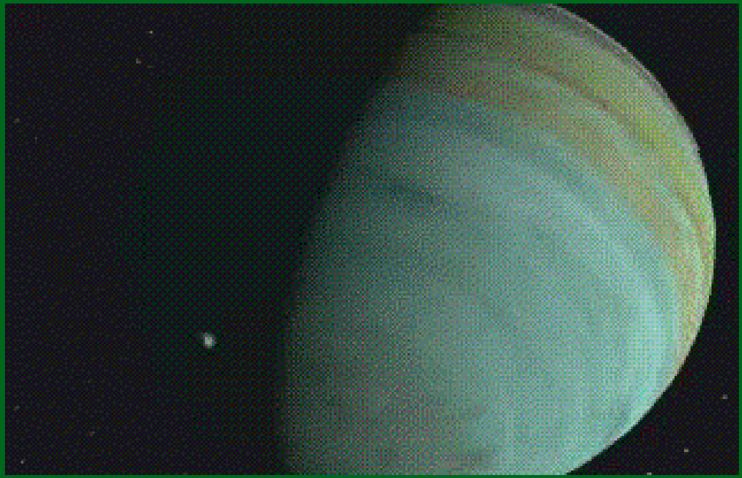


The construction is well known (e.g. Lockwood *A Book of Curves*) but this approach views it in another way.

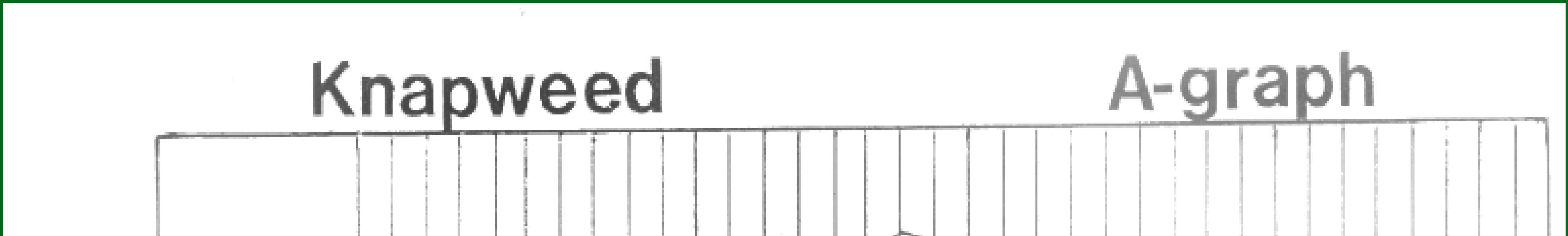
<<BACK

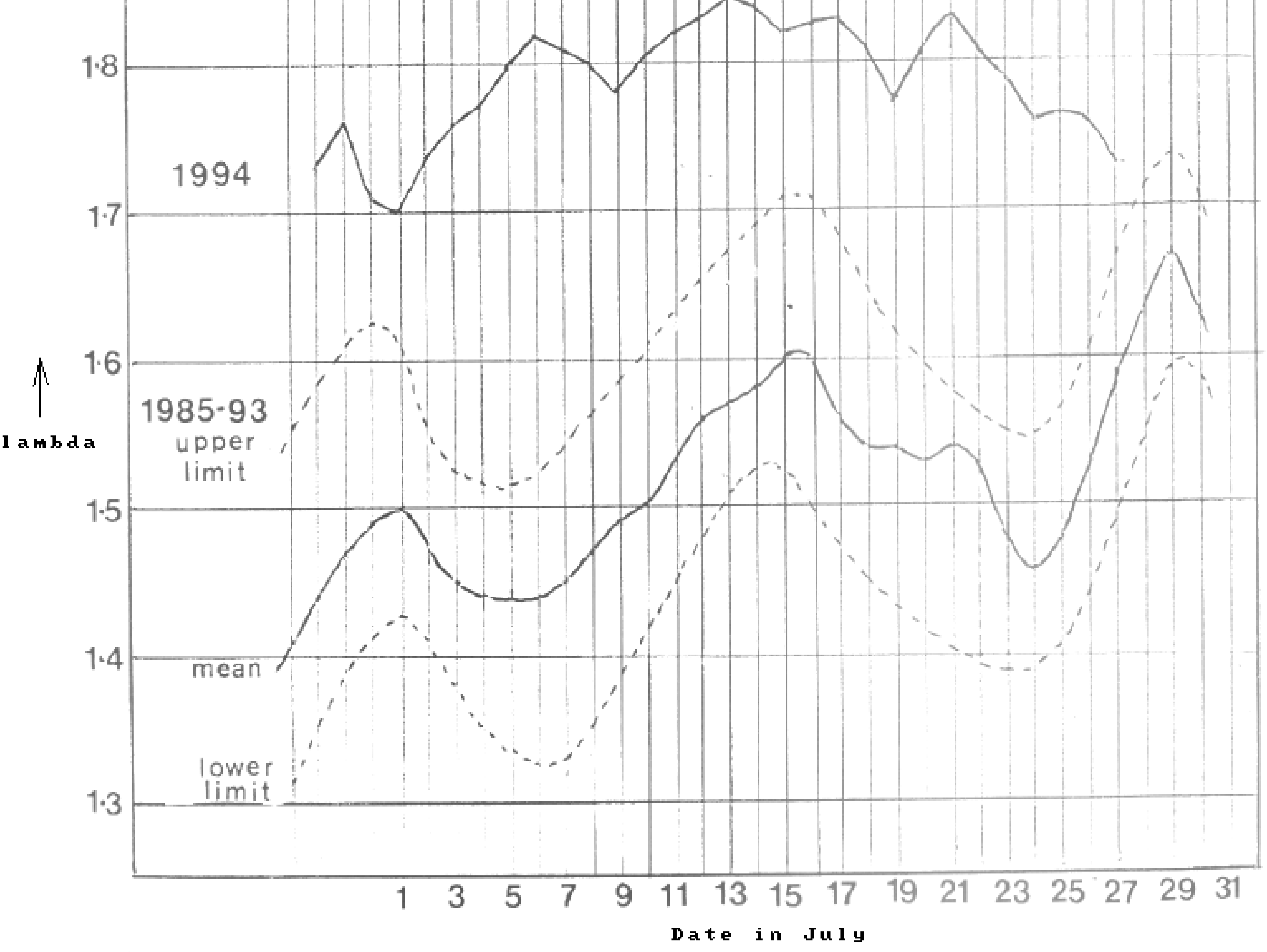
COMET IMPACT

Remember this?



It will be recalled that in 1994 Comet Shoemaker-Levy crashed into Jupiter. When Lawrence Edwards was investigating the profile shapes of leaf buds (i.e. their lambda values) he found that they vary rhythmically with a two-week cycle, except when in the neighbourhood of electric and/or magnetic fields. The original observation concerned a tree near a transformer, so to test the idea (and avoid waiting for trees to grow near transformers!) he checked the behaviour of knapweed (centaurea scabiosa & nigra) which could be checked under electric cables. The suppression of the two-weekly rhythm was indeed verified for such plants, but not if they were remote from cables. The two-week rhythm correlates with the conjunctions and oppositions of the Moon and a planet depending upon the tree. This is the first scientific evidence of a traditionally held relationship between trees and planets, now well verified by thousands of observations (predictably scoffed at by the *New Scientist* reviewer of Reference 7). In the case of knapweed the planet is Jupiter. By 1994 Edwards had made many observations of knapweed and so could compare its behaviour that year against the norm, and the result was interesting indeed, as illustrated below.



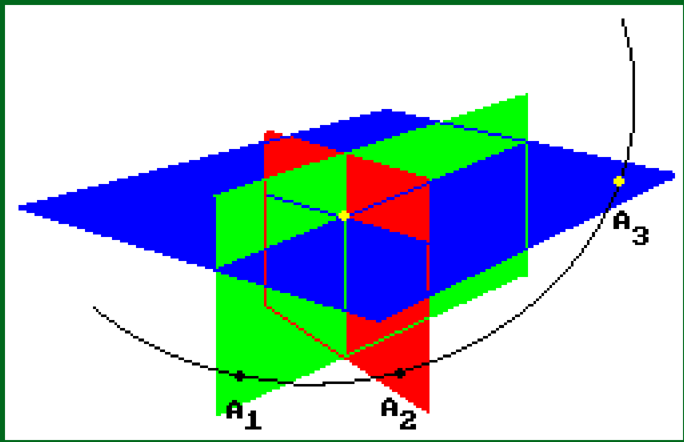

1·8
1·7
1·6
1·5
1·4
1·3
1994
1985-93
upper
limit
mean
lower
limit
lambda
1 3 5 7 9 11 13 15 17 19 21 23 25 27 29 31
Date in July

The upper and lower dotted lines show boundaries outside which no lambda values had ever been observed for this plant, before or since. The 1994 line shows values well outside these limits, suggesting that the comet impact affected the forms of the plant buds.

<<BACK

DOUBLE LINES

A path curve transformation (or space collineation) has an invariant tetrahedron with 6 invariant or double lines at least two of which must be real (see Path Curves). While this is quite easy to show algebraically, it is no trivial matter to derive a method of construction for the real double lines which is purely synthetic. For those who enjoy advanced pure projective geometry the proof and method is outlined below, and the full nine-page proof may be downloaded.



If A_1 A_2 A_3 are three successive corresponding points of a space collineation *C* then the bundles of planes in A_1 A_2 and A_3 are collinear. The triples of corresponding planes meet in points describing a cubic surface **C** (see e.g. *Semple and Kneebone "Algebraic Projective Geometry")* . **C** possesses in general six special lines each of which contains three corresponding planes, and which are thus double-lines of *C*. **C** intersects an arbitrary plane π in a plane cubic C_1 i.e. the points in π where triples of corresponding planes meet all lie on C_1. Each double-line of *C* must lie in all three planes of such a triple, so it must intersect C_1. If we consider the plane cubic C' in which **C** intersects the plane at infinity, generally a plane triple meeting in one of the points of C' is such that its planes meet in pairs in three parallel lines, and these will coincide for the double-lines. We select one line of each of these triples generated by the planes in A_2 and A_3 to intersect π in a second plane cubic C_2, which will generally intersect C_1 in 9 points. Six of these are significant and the lines through them are the double-lines of *C* e.g. if the two nodes coincide then so do 4 points of intersection leaving 5 others, giving 6 actual points. In these cases the three parallel lines of the plane triples must coincide as the triples also meet in π, so those six points give the double-lines of *C*. Since two plane cubics must meet in at least one real point we see that there is at least one real double-line of *C*. The double lines of *C* form the invariant tetrahedron we are seeking.

The reason for using plane cubics is that they are guaranteed to meet in at least one real point, unlike conics! From these ideas a method of construction (in principle, this is all in 3 dimensions) can be derived to find three of the double lines without having to construct anything more complicated than a conic. The other three require the additional construction of a plane cubic.

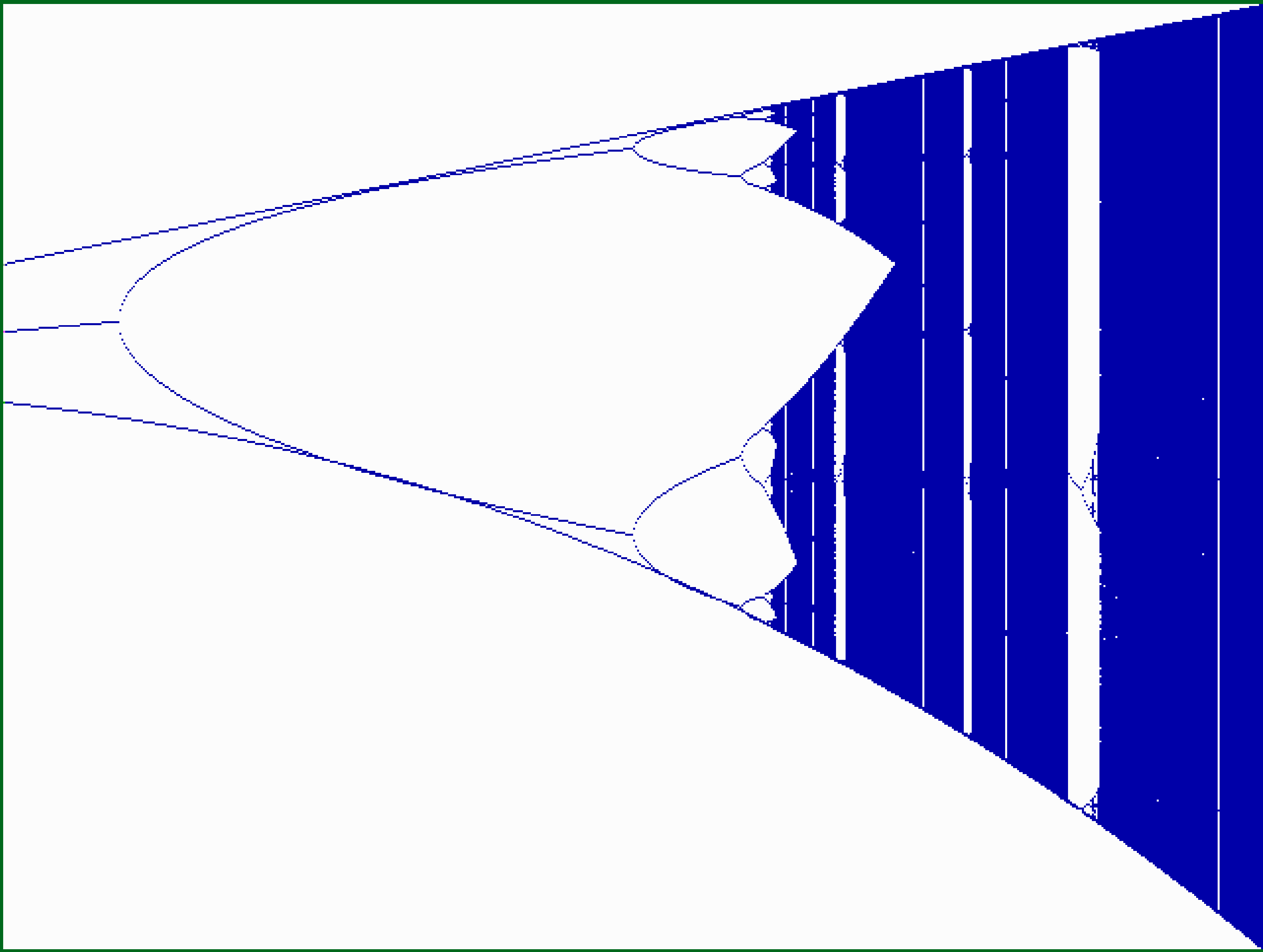
Download Proof

<<BACK

SIMPLE CHAOS THEORY

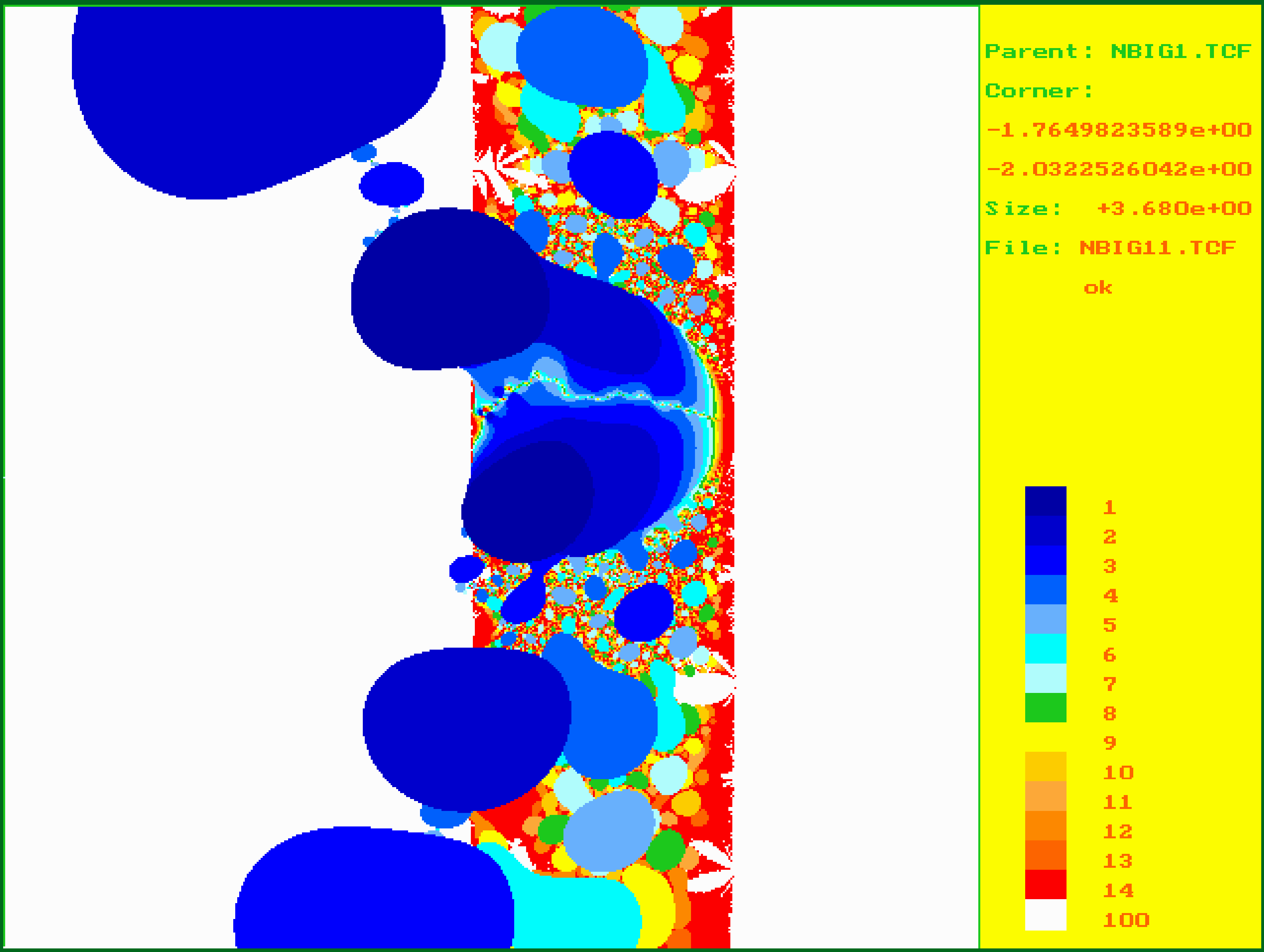
Can chaos be explained in a very fundamental way, without resorting to Hamiltonians and phase space, to give an intuitive feel for what is going on?   This is attempted here.

Chaos theory is to be found in many places from the giant red spot on Jupiter to dripping taps, and in the biological realm in heart fibrillation and brain seizures. Feigenbaum discovered a way of describing it, although he was not the first to discover chaos, it being known to Einstein, and even before him in the 19th Century from the study of dynamical systems where phase-space orbitals could cease to be well defined. It was largely ignored until the meteorologist Lorentz found that his simple model of the atmosphere did not give repeatable results. The advent of the PC with sufficient power to implement chaotic systems finally opened up the subject to wide research and application, although we might recall that Feigenbaum used a simple calculator to make his initial discovery! The actual existence of chaos as a fundamental fact rather than a mere appearance arising from inadequate precision in the calculations interested the engineer writing this. In other words he was sceptical: was it just 'hype'? What is actually happening is not easy to grasp from the advanced maths used. Below we show the classic figure for the equation $y=rx(1-x)$ when handled recursively i.e. the calculated value of y is re-inserted as x in the equation, and so on. The value of r is increased from 1 to 4 along the x-axis.

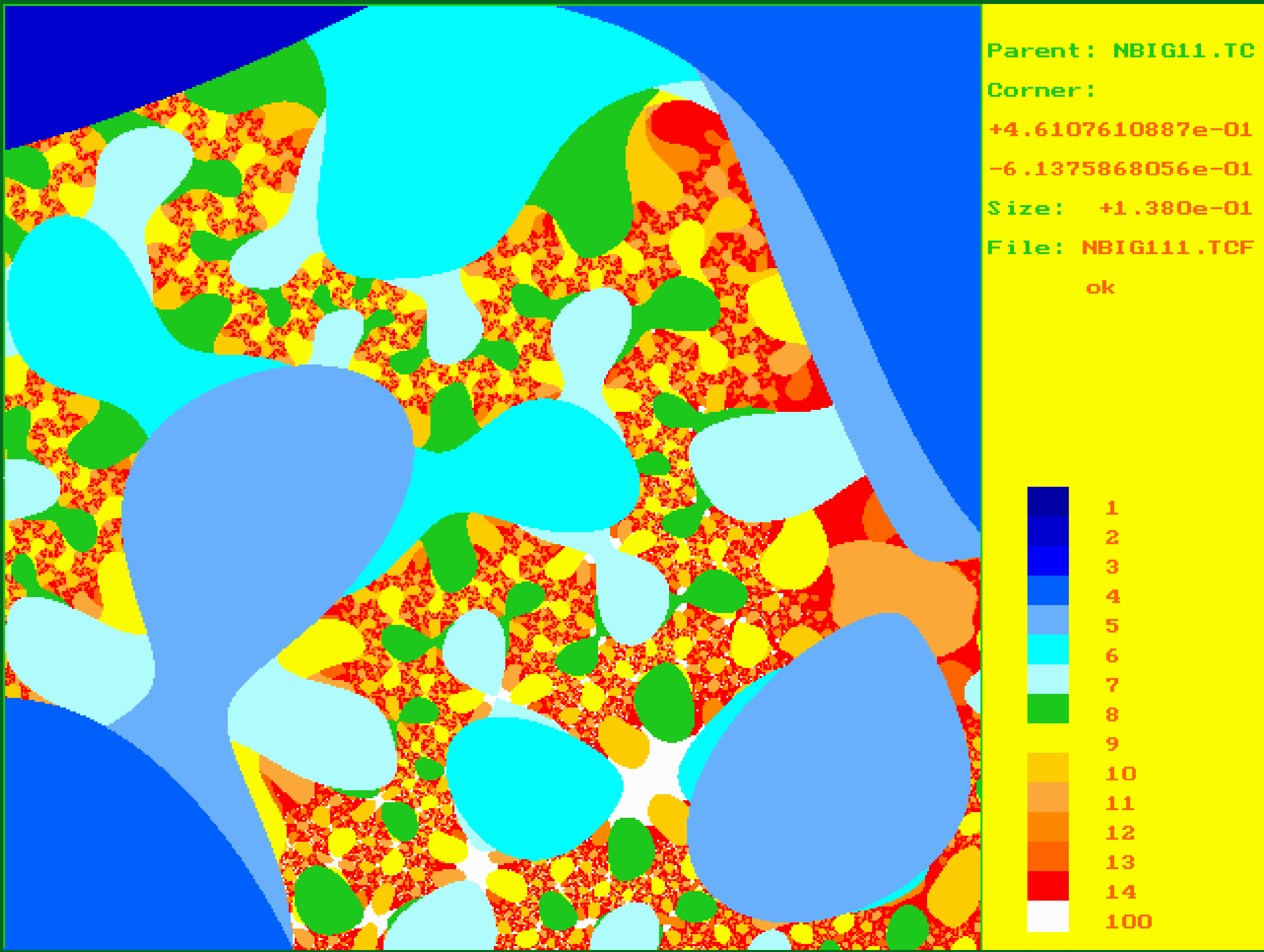


Ignoring the asymptotes, the function appears as a single line on the left where the recursions converge on a single value. As r is increased a bifurcation is reached at r = 3, the two resulting lines continuing toward the right until two more bifurcations occur (a so-called period-doubling) at r = 3.449, and so on.   The dense blue regions contain regions of genuine chaos mixed with reversions to non-chaos. In brief, what happens is that the interval between period doublings decreases as r increases, tending to zero before r reaches 4, at which point there are infinitely many bifurcations, and we have chaos. Reversion to non-chaos occurs when the equation cycles finitely for reasons we cannot explain briefly. An exploration of this together with a justification that chaos does exist fundamentally is explained in the article IS CHAOS GENUINE? which may be downloaded. It is a ZIP file containing three WORD files, one containing diagrams.

The tetrahedral complex is introduced in the archive article TETRAHEDRAL COMPLEX, , and it was found that chaos occurs within projective geometry itself when polarity is traversed recursively in a tetrahedral complex. The picture below shows a diagram for this chaotic polarity which is its equivalent of the famous Mandelbrot set.



The colour codes for the number of iterations before the cubic function goes to infinity are shown on the right. This is only a portion of the whole set which extends to infinity. On the upper left there is a 'fractal bridge' between two 'globs', which looks the same at all magnifications, reminiscent of God reaching his finger towards Adam. The true fractal nature of the process is illustrated by the following picture taken from within the vertical strip:



The equation relating polar lines in the complex which when iterated leads to the above pictures is

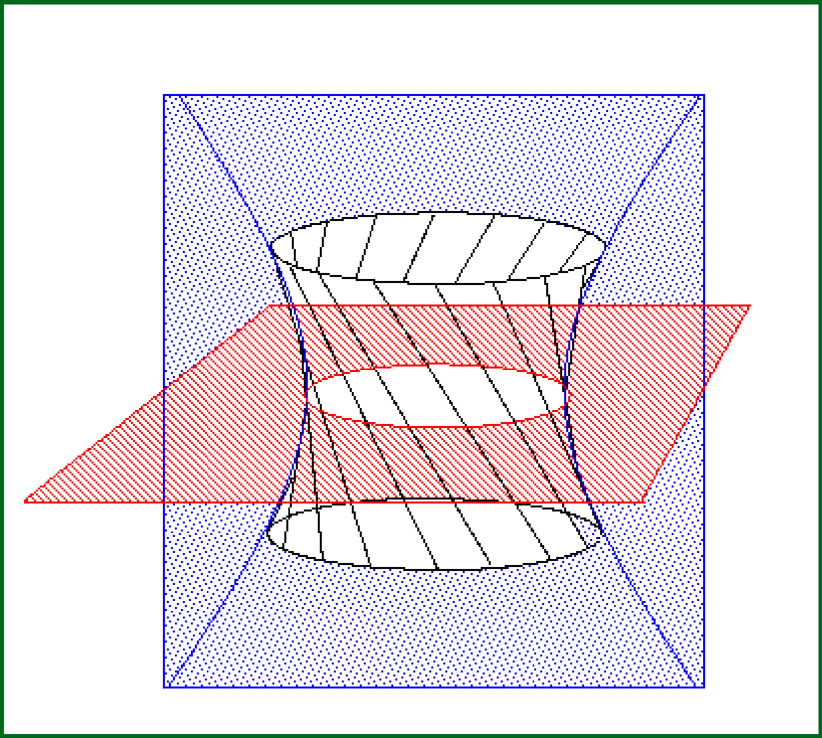
$$\lambda' = \frac{\left(k^2 - 2k + \lambda\right)^2}{\left(k^2 - 2k\lambda + \lambda\right)^2}\lambda$$

where lambda is the cross-ratio in which a line cuts the tetrahedron and k is the fundamental cross-ratio defining the complex.

ASYMPTOTIC LINES

George Adams was interested in *asymptotic lines* as possible interfaces between physical and ethereal forces. In terms of counterspace this might be equivalent to a linkage between space and counterspace.

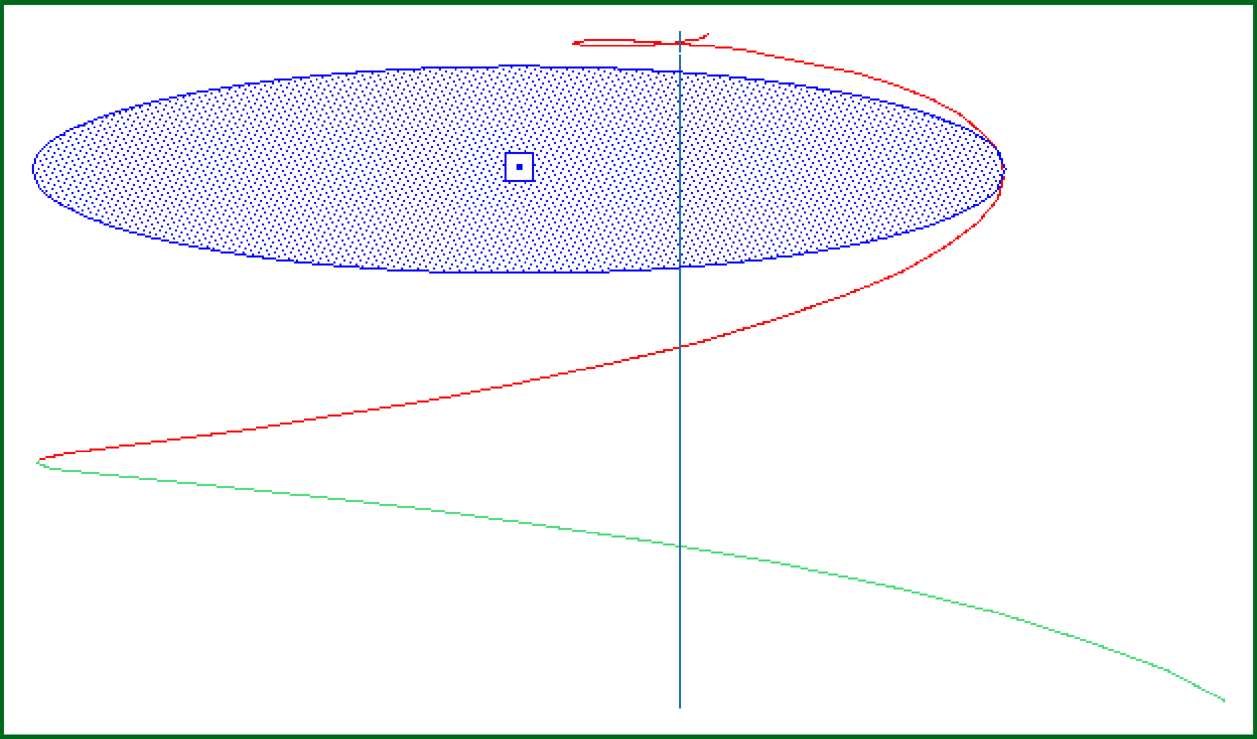
An asymptotic line is a kind of boundary between positive and negative curvature on a surface. For example, consider a ruled hyperboloid:



The red plane intersects it in a circle, a curve which has positive curvature, while the blue plane intersects it in a hyperbola, which has negative curvature. If we rotate the plane from red to blue, at one position it meets the surface in two straight lines called *rulers*, which have infinite (or no) curvature. Those lines are asymptotic lines because they mark the transition between cross sections with positive and negative curvature. There are many asymptotic lines on a surface, and the rulers are the asymptotic lines in this case.

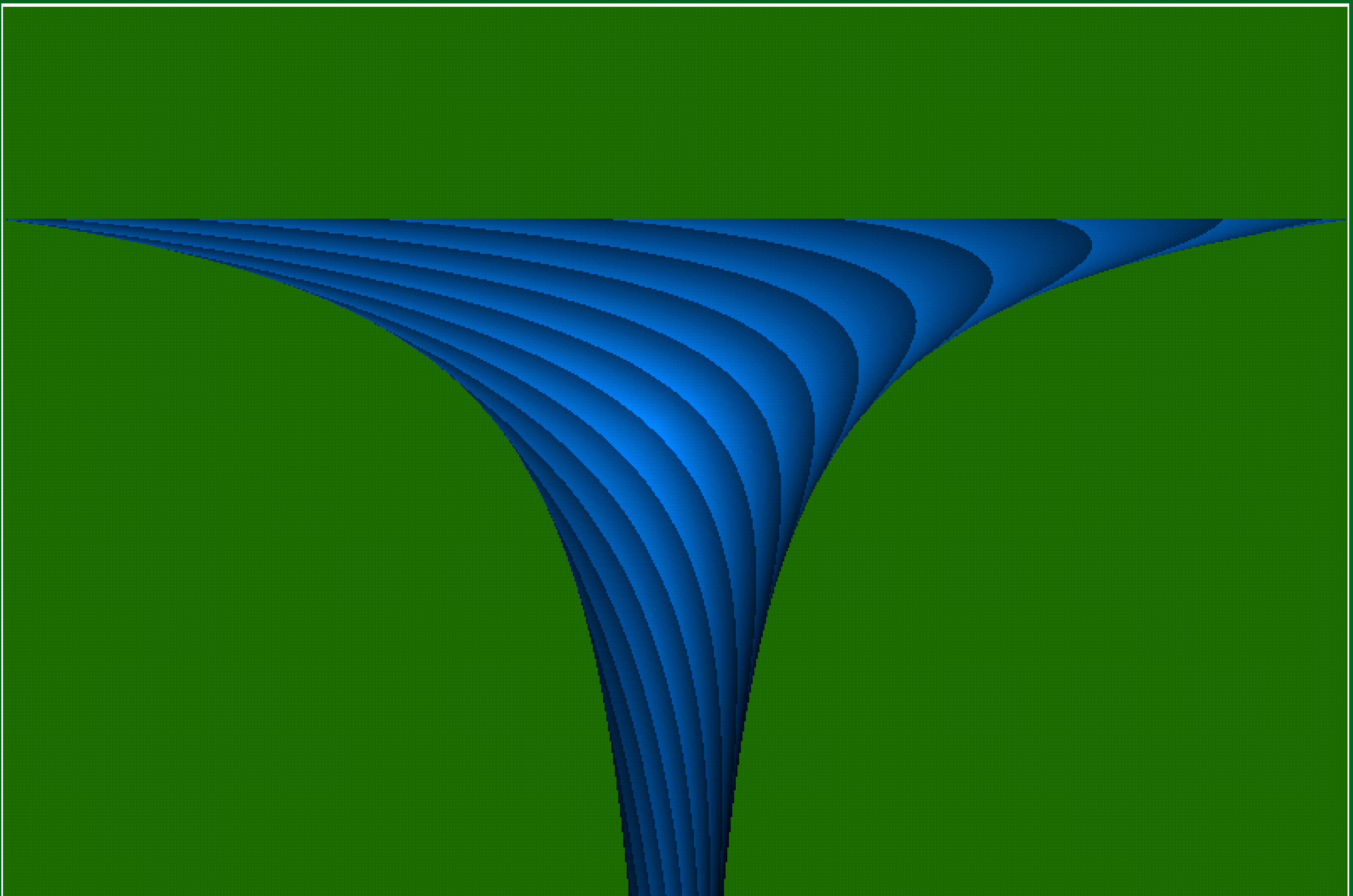
It is clear that no such argument can be applied to an ellipsoid as all intersecting planes meet it in ellipses, which have positive curvature. Surfaces such as the ruled hyperboloid are said to have negative curvature because planes can meet them in curves with either positive or negative curvature, and only such surfaces can have asymptotic lines.

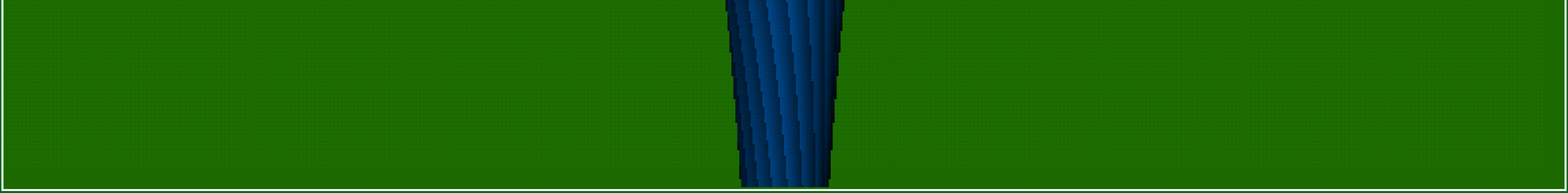
Another way of expressing all this is to say that curves with positive curvature have their *centres of curvature* inside the surface, while those with negative curvature have their centres of curvature outside. The asymptotic curves are a transition between these two cases. For a circle the centre of curvature is obviously its centre, while for other curves it varies and at a given point it is the centre of the tangential circle in the osculating plane which has the same curvature as the curve at that point.



For more complex surfaces there may exist points such that all the curves through them have their centres of curvature on only one side of the surface, known as *elliptical points*, and *hyperbolic points* with centres of curvature on both sides for the various curves passing through it. A surface must possess hyperbolic points for it to contain asymptotic lines.

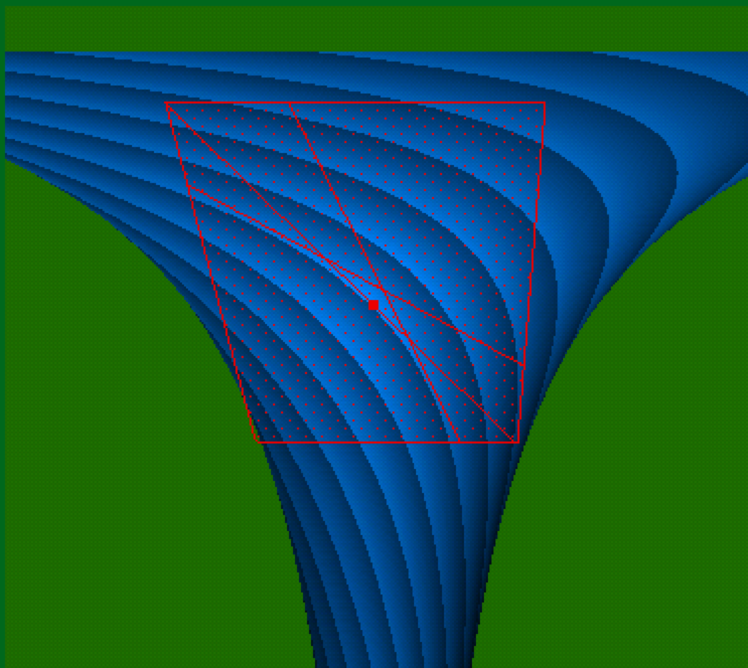
The spiralling curves on the vortex below are its asymptotic lines:





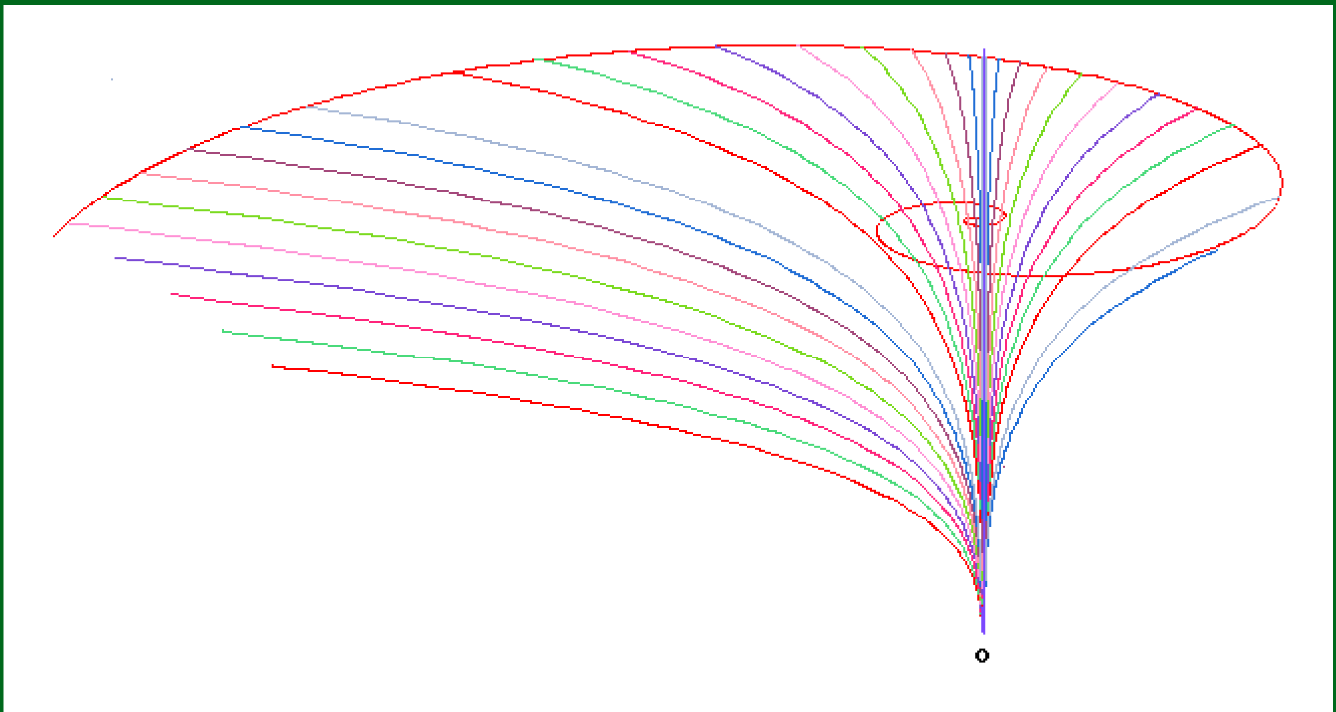
For such surfaces we have to go to the infinitesimal and consider the *asymptotic direction* at a point P on the surface. If we take all possible planes in that point each meets the surface in a curve, and the asymptotic direction is that tangent at P which separates tangents to curves with positive curvature from those with negative curvature. In general there exist two asymptotic directions through P, tangential to two curves such that the tangent at every point of them is an asymptotic direction, and hence those two curves are asymptotic lines of the surface.

Furthermore, the asymptotic lines are curves whose osculating planes coincide with the tangent planes at each point of the curve. Now an osculating plane at a point P on a curve is that plane in which the tangent at P is momentarily turning i.e. the curve momentarily lies in that plane. In the diagram below the red plane represents a tangent plane and the three tangents illustrate what is meant, although they should of course be 'consecutive' tangents as the curve only lies in the plane at the indicated point of tangency:

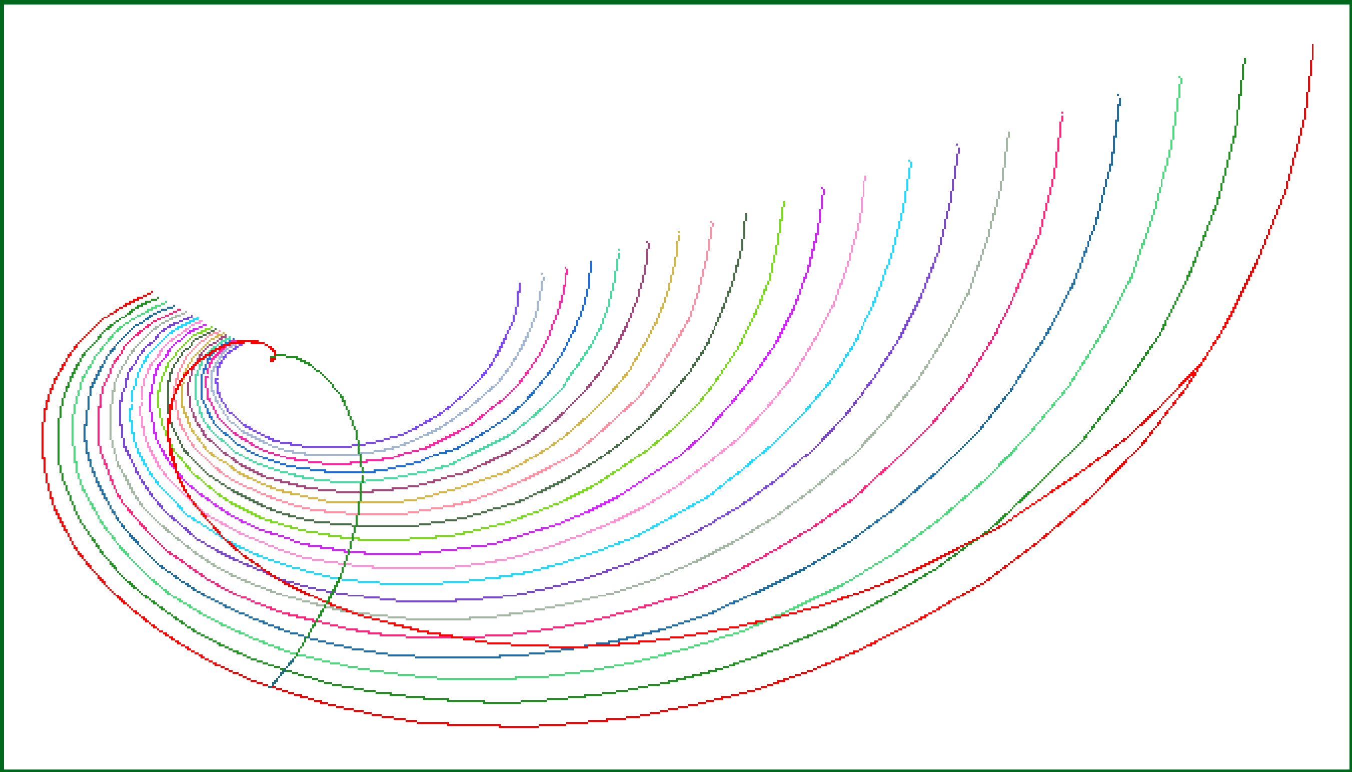


If that plane is also tangential to the surface then the curve is neither turning towards the inside of the surface nor towards its outside, and hence demarcates those curves with their centres of curvature on one side of the surface from those with them on the other. Hence the tangent is an asymptotic direction.

Returning to the vortex shown above, it is a surface defined by path curves with lambda equal to -1.618. There can be many sets of path curves on one such surface, and in fact its asymptotic curves are also path curves. This is not a trivial result, and a little manipulation is required to prove it. Given a path curve specified by its *lambda* and *epsilon,* a unique surface exists for which it is an asymptotic line, and all other such asymptotic lines on that surface with the same sense have the same parameters. The parameters of the surface are *mu* and *beta*, where mu is the lambda value for its vertical profiles (always negative) and beta is the cotangent of the angle defining the horizontal logarithmic spiral cross sections. Thus given such a spiral together with a given point O below the centre of the spiral, the surface may be envisaged as composed of all the vertical path curves specified by mu which start from O and intersect the spiral.



If beta = 0 then the spirals degenerate to circles, which is the case for the previous vortex. In that case the asymptotic lines in the opposite sense have the same lambda but their epsilon is reversed in sign i.e. they are essentially the same path curves but winding round the surface in the opposite sense. mu equals lambda in that case, and it is epsilon that singles out the asymptotic path curves from all the others. The previous vortex has epsilon = 0.2429. For more general surfaces such as above the second set of asymptotic path curves has a different lambda from the first. Another way of thinking of the surface is to take a fixed vertical path curve and the axis, and rotate a horizontal logarithmic spiral such that its centre remains on the axis and it always intersects the path curve; its plane will move upwards or downwards parallel to itself. The diagram below shows a set of logarithmic spirals for such a surface seen from the top, with an example of each type of asymptotic line:



The practical formulae for calculating the asymptotic path curves derived by the author are:

$$\beta = \frac{(\lambda+1)^2 + 4\lambda\epsilon^2}{2\epsilon(\lambda^2 - 1)}$$

$$\mu = -\frac{4\epsilon^2\lambda^2 + (\lambda+1)^2}{4\epsilon^2 + (\lambda+1)^2}$$

$$\epsilon^2 = -\frac{(1+\beta^2)(1+\mu^2)}{4\mu}$$

$$\lambda = \frac{\mu \pm \beta(1+\mu)\sqrt{[-\mu(1+\beta^2)]}}{1+\beta^2(1+\mu)}$$

It is clear that although a unique surface is defined by choice of lambda and epsilon for the asymptotic line, a given surface has in general two possible values each of epsilon and lambda, corresponding to the two sets of asymptotic lines winding in opposite senses (note that epsilon is the same for both apart from sign, but lambda is distinct). Also, setting beta = 0 makes lambda unique and equal to mu, as stated above.

The late Dr Georg Unger first analysed asymptotic path curves and derived the following formula for George Adams:

$$\cot\tau = \tfrac{1}{2}\{(1+\epsilon^2)/\alpha - \alpha)\}$$

where τ = equiangular spiral parameter in formula $r = r_0 e^{\theta \cot\tau}$
so $\cot\tau = \beta$
α = pathcurve parameter = $\epsilon(\lambda-1)/(\lambda+1)$

Footnote:

The great mathematician Gauss studied curves in surfaces when commissioned to make a survey of Germany, and derived the equations for such curves and their curvatures. This is the subject matter of *differential geometry*, and several types of curvature are defined. A very beautiful theorem is that of *Meusnier* which states that the circles of curvature of all plane sections through the same line element of a surface lie on a sphere. There are two *principal radii of curvature* at a point, R_1 and R_2 , obtained by solving the equation for coincidence of the two directions of curvature for a normal section. The tangents for R_1 and R_2 define the *principal directions*, and in general two curves through a point are such that all their tangents are principal directions. Such curves are called *lines of curvature*. The *Gaussian curvature* is defined as $K=1/R_1R_2$, and the *average curvature* H by $2H=1/R_1+1/R_2$. The angles between two asymptotic lines through a point are bisected by the lines of curvature through that point. Texts on vector algebra or differential geometry derive these results.

Volatile

COVARIANCE AND CONTRAVARIANCE

When studying tensor calculus the distinction between covariance and contravariance may be obscure and is rarely explained visually. A geometric explanation will be exhibited here.

First we will explain the distinction between the covariant and contravariant components of vectors, thinking of vector-fields where a vector is defined at a point rather than as a position vector. This extends naturally to the components of higher order tensors. Strictly speaking, despite usage to the contrary, there is no such thing as a “covariant vector” or a “contravariant vector”. A vector is a vector is a vector. However it may be handled in two ways. Firstly by means of its components parallel to the coordinate directions which form a parallelogram in the two-dimensional case, in the same way that *dx* and *dy* are defined as the sides of the parallelogram related to an infinitesimal displacement *ds*. These components are referred to as its *contravariant components*. Secondly we may handle it by means of its resolved parts along the coordinate directions, which are its *covariant components*. The latter are the inner products of the vector with the coordinate unit vectors. The distinction is important e.g. when finding inner products such as ***F.s*** for the work done by a force ***F*** producing a displacement ***s***. We will follow that up later.

We will work with vectors in two dimensions to illustrate the principles involved. We will use non-orthogonal cartesian coordinates i.e. coordinates defined relative to non-orthogonal axes. However tensors are especially concerned with the use of curvilinear coordinates, where vectors and tensors are referred to curved coordinate lines which approach linearity at infinitesimal distances. The coordinate axes used below should be regarded as the tangents to such coordinate lines in such cases, and vectors as directed magnitudes at an origin O which is a local point in a field. The coordinate directions thus vary as O is varied. This covers cases where both coordinates are of the same type (polar coordinates in two dimensions are an example where they are not).

curvilinear coordinates

Figure 1

Contravariant Components

The components of a vector in two dimensions are defined in the literature in relation to a change of coordinates from (x,y) to (x',y'), say. The contravariant components are those which transform as follows e.g. for the new coordinate x' in terms of the old (x,y):

Tensor_1 (1)

and similarly for y'. This is far from obvious at first sight, so we will show how the partial derivatives relate to the geometry.
This is how the coordinates themselves are transformed, and oddly enough vectors defined in this way are referred to as *contravariant*, which at first sight seems rather perverse. However the comments about inner products below may shed light on this oddity.

Contravariant image

Figure 2

The vector at O is represented by OV and the parallelogram-component on the axis OX is OA, where VA is parallel to the axis OY. We will only illustrate the situation for the x-components. If we change coordinates to OX', OY' then the new x-component is OA' where VA' is parallel to OY'. Now we join A to P on OX' such that AP is parallel to OY'. Using the sine rule we get

Equation 2 (2)

where $\gamma=\phi+\beta-\alpha$ and •=180-ϕ-β.

Noting that OA'=*x'*, OA=*x* and AV=*y*, partial differentiation of this with respect to *x* gives

Equation 2a from triangle OAP, holding *y* constant, and

Equation 2b from triangle VQA', holding *x* constant,

giving from (2)

Equation 2c

as required. A similar argument holds for the new *y* coordinate. The generalised version of (1) for more than two dimensions, using overlines instead of primes, is

Equation 2d

or, using the repeated-index summing convention for k,

Equation 3 (3)

For the contravariant components it is customary to use superscripts for the indices such as j and k.
Thus our previous $x' = x^1$ and $y'=x^2$.

Useful expressions for the contravariant coordinates of OV are, using the sine rule,

Equation 4 (4)

Covariant Components

The covariant components of a vector are defined by the transformation

Equation 5 (5)

using subscripts for the indices in the covariant case. For the x-coordinate in two dimensions this is

Equation 6 (6)

where the partial derivatives are "inverted" compared with the contravariant case. We start by assuming we know x, y, α and ϕ i.e. we know the initial coordinates of the vector rather than its magnitude OV=v or its angle θ to OX. OA=x and OB=y (Figure 3):

Figure 3

Figure 3

Then

equation 7 (7)

Solving for θ gives

Equation 8 (8)

Now

Equation 8a

which by (7) is

Equation 8b

which by (8) is

Equation 9

We now encounter a subtlety of the meaning of the "inverted" partial derivatives, for they refer to the coordinates which are contravariant, so we must relate this back to them as follows:

Figure 4

Figure 4

If OX'=δx', OX=δx and OY=δy then using the sine rule in the infinitesimal case we get

Equation 9a

showing that (9) is the same as (6), as required. For more than two dimensions the principle is the same but OV is no longer necessarily in a coordinate plane.

We have thus exhibited how the geometrical interpretation of covariance and contravariance relates to the formal definitions when the components are of the same type.

Inner Product

The distinction between contravariance and covariance is important e.g. when finding inner products such as ***F*.*s*** for the work ***W*** done by a force ***F*** producing a displacement ***s***. We take the inner product of the two vectors which usually means resolving ***F*** along the direction of ***s***. The actual evaluation of ***W*** amounts to summing the products of the coordinate-system-components of ***s*** by the resolved parts of ***F***. That is, we sum the products of the contravariant components of ***s*** and the covariant components of ***F*** as for an inner vector product. To use instead the contravariant components of ***F*** (which are perfectly respectable quantities) would obviously give the wrong result for ***W***. However, we may instead use the covariant components of ***s*** multiplied by the contravariant ones of ***F*** and get the correct result, but it seems an unnatural way to handle the problem. It is more natural to handle ***F*** by means of its covariant components, which is perhaps why the loose description of a force as a "covariant vector" has crept in. Similarly ***s*** is most naturally handled by means of its parallelogram components.

We will now show how this works explicitly. Applying (4) to the vector ***s*** represented by OV of length *s* as in Figure 2, but at an angle ψ to OX, gives

Equation 9b

The covariant components of ***F*** represented by OV as in Figure 3 are:

Equation 9c

and combining the two gives the inner product in tensor form:

Equation 9c

which is the standard expression for the inner product.

If we change the coordinate system then the covariant components of ***F*** will change such that the above inner product remains invariant (and valid!). This may explain the use of *covariant* for such components.

Generally a tensor is characterised by a set of functions defining how its components vary with the coordinates. A set of functions comprise a tensor if the components satisfy (3) or (5). Another test is to multiply a set of functions by a tensor, and if the result is a tensor then so are those functions. To find out whether the functions are the simplest possible for a tensor is more difficult, remembering that the tensor is an entity that is described by the functions, just as a velocity is an independent physical entity that may be described in various ways. Such an entity exists independently of the coordinates used to describe it since any equations involving it will, in view of (3) and (5), be the same in any coordinate system e.g. work done expressed by an inner product. However the functions may prove to be simpler in one coordinate system than another e.g. a radial electric field is better described in polar coordinates than cartesian.

PIVOT TRANSFORMS

ANNEX 1

TANGENT PLANES TO SURFACES

If a surface is defined by two parameters u,v i.e.

$$x=x(u,v)$$
$$y=y(u,v)$$
$$z=z(u,v)$$

and (x,y,z) is the tangent point of a plane while (ξ, η, ζ) are the coordinates of a variable point in that plane then the plane is given by the equation

$$(\xi-x)\frac{\partial(y,z)}{\partial(u,v)}+(\eta-y)\frac{\partial(z,x)}{\partial(u,v)}+(\zeta-z)\frac{\partial(x,y)}{\partial(u,v)}=0$$

where e.g.

$$\frac{\partial(y,z)}{\partial(u,v)}\equiv\begin{vmatrix}\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v}\end{vmatrix}$$

(c.f. for example *Partial Differentiation* by R.P. Gillespie).

If (x,y,z)=(0,0,0) then

$$\xi\frac{\partial(y,z)}{\partial(u,v)}+\eta\frac{\partial(z,x)}{\partial(u,v)}+\zeta\frac{\partial(x,y)}{\partial(u,v)}=0$$

which gives a plane through the origin parallel to the tangent plane with the determinants as its plane coordinates.

It follows that the direction cosines of the normal to the surface at (x,y,z) are proportional to

$$\left\{\frac{\partial(y,z)}{\partial(u,v)},\frac{\partial(z,x)}{\partial(u,v)},\frac{\partial(x,y)}{\partial(u,v)}\right\}$$

We select the parameters u,v as follows

u = the vertical height of G above O (see main text)
v = the height h of the contour plane.

Then from (13) in the main text we have

$$y(u,v)=\frac{kv(y_3-mx_3)}{1+m^2}$$
$$x(u,v)=-my(u,v)$$
$$z(u,v)=v$$

noting that a+b=e+c=1 (c.f. Figure 9).

Since b=h' and a=1-b, and recalling (11) and (12), we have

$$b=\frac{k_1 v}{k_1 v+k_2(1-v)}$$
$$a=\frac{k_2(1-v)}{k_1 v+k_2(1-v)}$$
$$x_3=\frac{a\left[ex_2+u(x_1-x_2)\right]}{e\left[a-e+u\right]}$$
$$y_3=\frac{a\left[ey_2+u(y_1-y_2)\right]}{e\left[a-e+u\right]}$$

The following partial derivatives are then obtained:

$$\frac{\partial y_3}{\partial u}=\frac{a\left[(a-e)(y_1-y_2)-ey_2\right]}{e(a-e+u)^2}$$
$$\frac{\partial x_3}{\partial u}=\frac{a\left[(a-e)(x_1-x_2)-ex_2\right]}{e(a-e+u)^2}$$
$$\frac{\partial y_3}{\partial v}=-\frac{k_2 b^2}{k_1 ev^2}\frac{(u-e)\left[ey_2+u(y_1-y_2)\right]}{(a-e+u)^2}$$
$$\frac{\partial x_3}{\partial v}=-\frac{k_2 b^2}{k_1 ev^2}\frac{(u-e)\left[ex_2+u(x_1-x_2)\right]}{(a-e+u)^2}$$

Since z=v we have

$$\frac{\partial z}{\partial u}=0$$
$$\frac{\partial z}{\partial v}=1$$

We now require ∂m/∂u and ∂m/∂v which requires us to find a suitable expression for m. Referring to Figure 9 let the equation of the tangent to the circle be y=mx+q. Then since it contains (x_3,y_3) we have $y_3 = mx_3+q$.

It intersects the circle $x^2+y^2-2xx_1-2yy_1+(x_1^2+y_1^2-R^2)=0$ in the points given by

$$(mx+q)^2+x^2-2xx_1-2y_1(mx+q)+(x_1^2+y_1^2-R^2)=0$$

which may be rearranged as a quadratic equation in x:

$$x^2(m^2+1)+2x(mq-y_1m-x_1)+(q^2-2y_1q+x_1^2+y_1^2-R^2)=0$$

Setting the descriminant to zero to give us equal roots (for a tangent) we find

$$m^2(R^2-x_1^2)+2mx_1(y_1-q)+(R^2+2qy_1-y_1^2-q^2)=0$$

Substituting $q=y_3-mx_3$ and simplifying gives

$$m^2[R^2-(x_1-x_3)^2]+2m(x_1-x_3)(y_1-y_3)+R^2-(y_1-y_3)^2=0$$

which is a quadratic equation in m for the two tangents to the circle in Figure 9, in terms of known quantities. Noting that x_1 and y_1 are constant we obtain from this

$$\frac{\partial m}{\partial u}=\frac{m\left[(x_1-x_3)\frac{\partial y_3}{\partial u}+(y_1-y_3)\frac{\partial x_3}{\partial u}\right]-m^2\left[(x_1-x_3)\frac{\partial x_3}{\partial u}+R\frac{\partial R}{\partial u}\right]-(y_1-y_3)\frac{\partial y_3}{\partial u}-R\frac{\partial R}{\partial u}}{m\left[R^2-(x_1-x_3)^2\right]+(x_1-x_3)(y_1-y_3)}$$

Similarly we get

$$\frac{\partial m}{\partial v}=\frac{m\left[(x_1-x_3)\frac{\partial y_3}{\partial v}+(y_1-y_3)\frac{\partial x_3}{\partial v}\right]-m^2\left[(x_1-x_3)\frac{\partial x_3}{\partial v}\right]-(y_1-y_3)\frac{\partial y_3}{\partial v}}{m\left[R^2-(x_1-x_3)^2\right]+(x_1-x_3)(y_1-y_3)}$$

All the subsidiary partial derivatives are given above except ∂R/∂u, which needs to be derived from an expression for R which is independent of m, and determined by the vortex.

In the main text we saw that R is given by

$$R=W\frac{e-u}{u}\left(\frac{u}{e}\right)^{\mu}$$

so clearly

$$\frac{\partial R}{\partial v}=0$$

and differentiating wrt u gives

$$\frac{\partial R}{\partial u}=\frac{W}{u^2}\left(\frac{u}{e}\right)^{\mu}\left[\mu(e-u)-e\right]$$

Now we can find the partial derivatives of x, y and z wrt u and v:

$$\frac{\partial y}{\partial u}=\frac{\partial}{\partial u}\left(\frac{kv(y_3-mx_3)}{1+m^2}\right)$$

$$=\frac{kv}{1+m^2}\left\{\frac{\partial y_3}{\partial u}-m\frac{\partial x_3}{\partial u}-x_3\frac{\partial m}{\partial u}-\frac{2\mathrm{m}(y_3-mx_3)}{1+m^2}\frac{\partial m}{\partial u}\right\}$$

$$\frac{\partial y}{\partial v}=\frac{kv}{1+m^2}\left\{\frac{\partial y_3}{\partial v}-m\frac{\partial x_3}{\partial v}-x_3\frac{\partial m}{\partial v}-\frac{2\mathrm{m}(y_3-mx_3)}{1+m^2}\frac{\partial m}{\partial v}\right\}+\frac{k(y_3-mx_3)}{1+m^2}$$

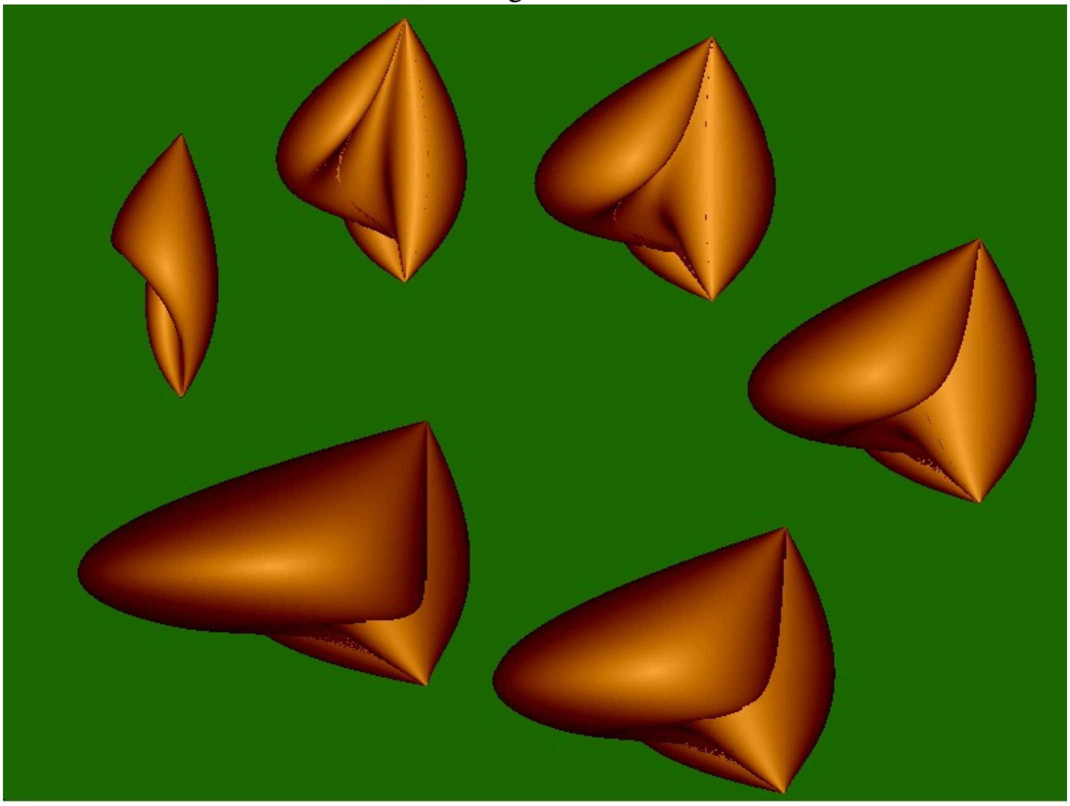
$$\frac{\partial x}{\partial u}=\frac{\partial(-my)}{\partial u}=-m\frac{\partial y}{\partial u}-y\frac{\partial m}{\partial u}$$

$$\frac{\partial x}{\partial v}=-m\frac{\partial y}{\partial v}-y\frac{\partial m}{\partial v}$$

This gives us what we need to calculate

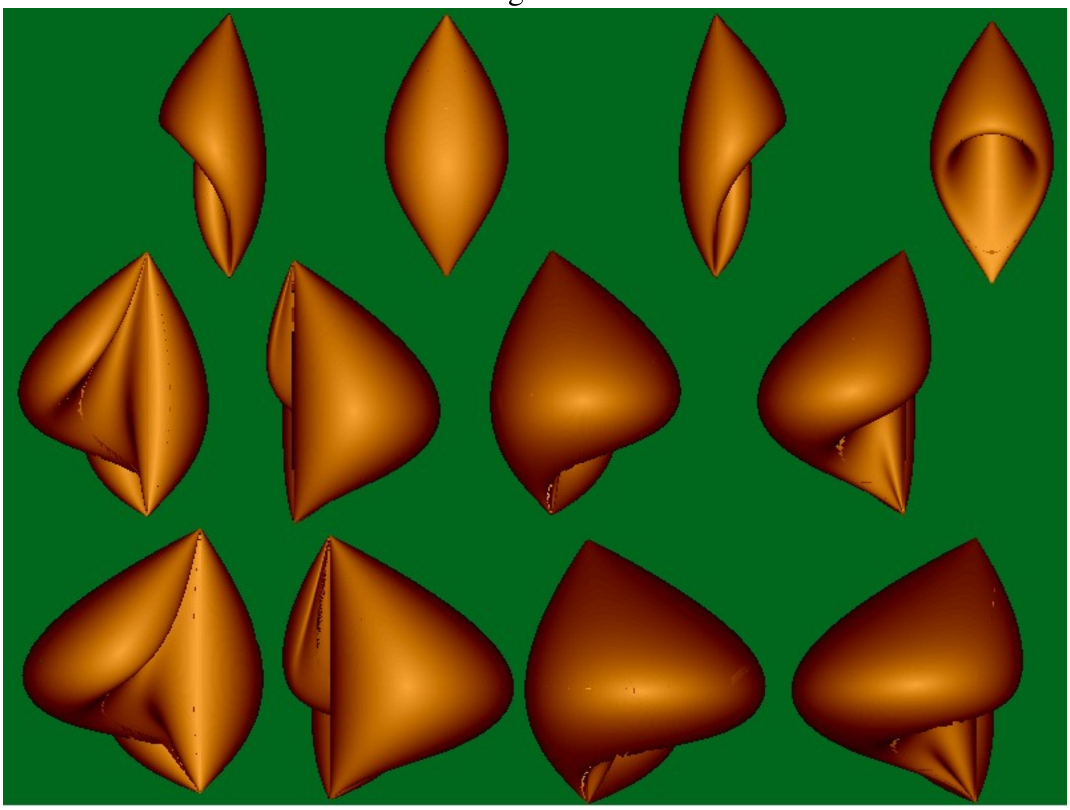
$$\frac{\partial(y,z)}{\partial(u,v)},\frac{\partial(z,x)}{\partial(u,v)}\text{ and }\frac{\partial(x,y)}{\partial(u,v)}$$

Figure 1



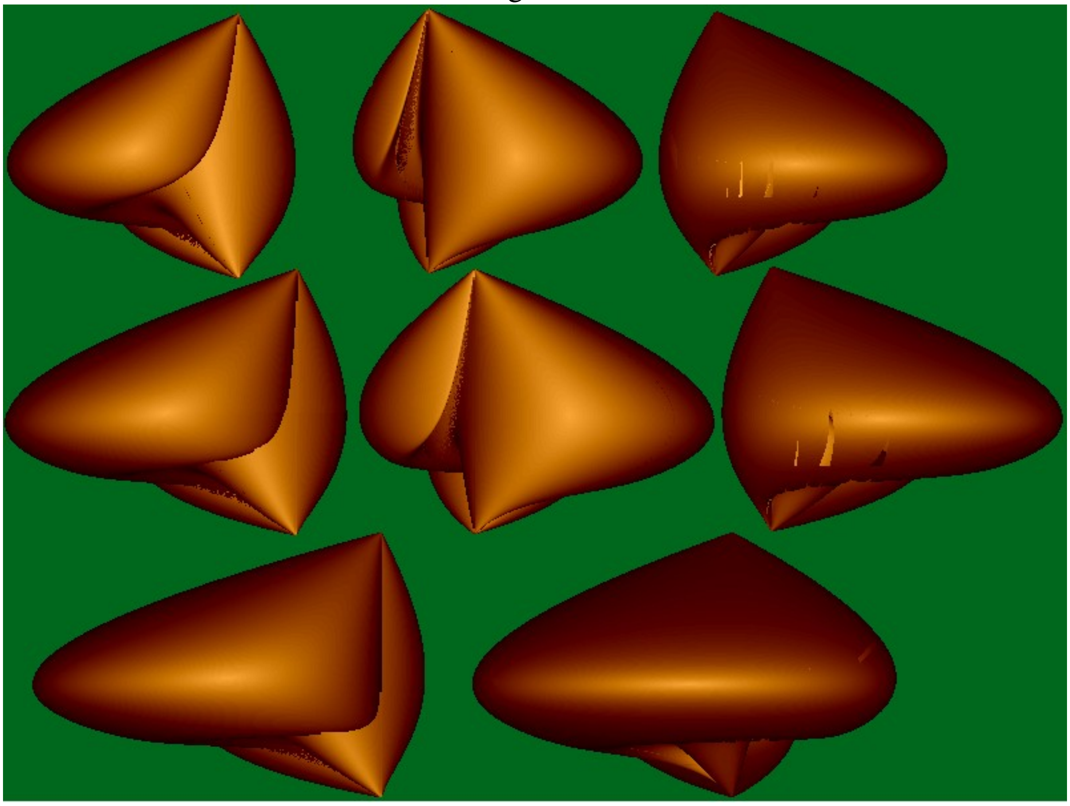
Three dimensional versions of Figure 8

Figure 2



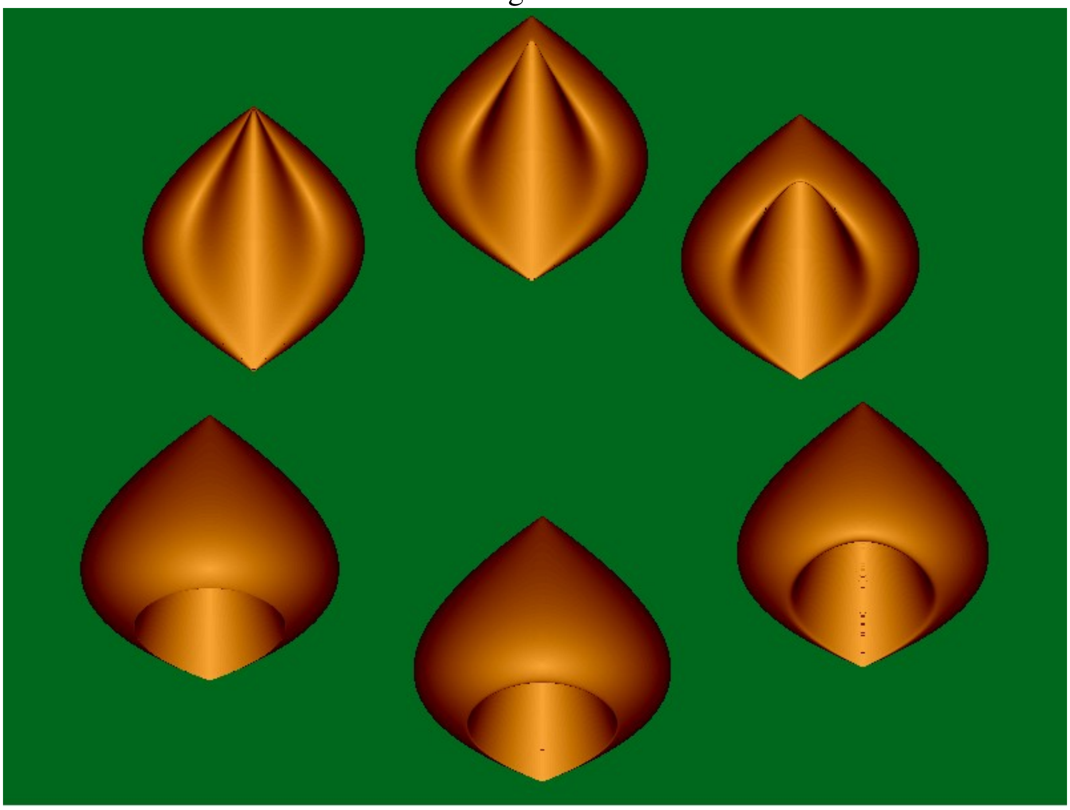
First three of above cases from various views

Figure 3



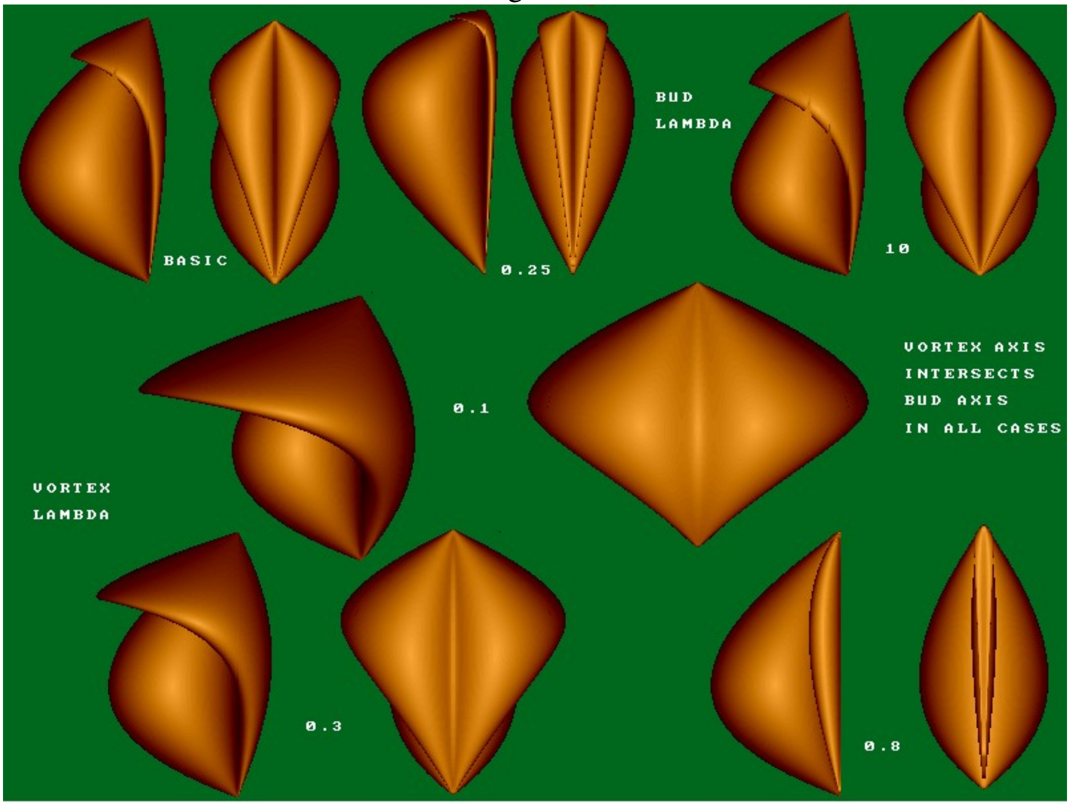
Last three of above cases from various views

Figure 4



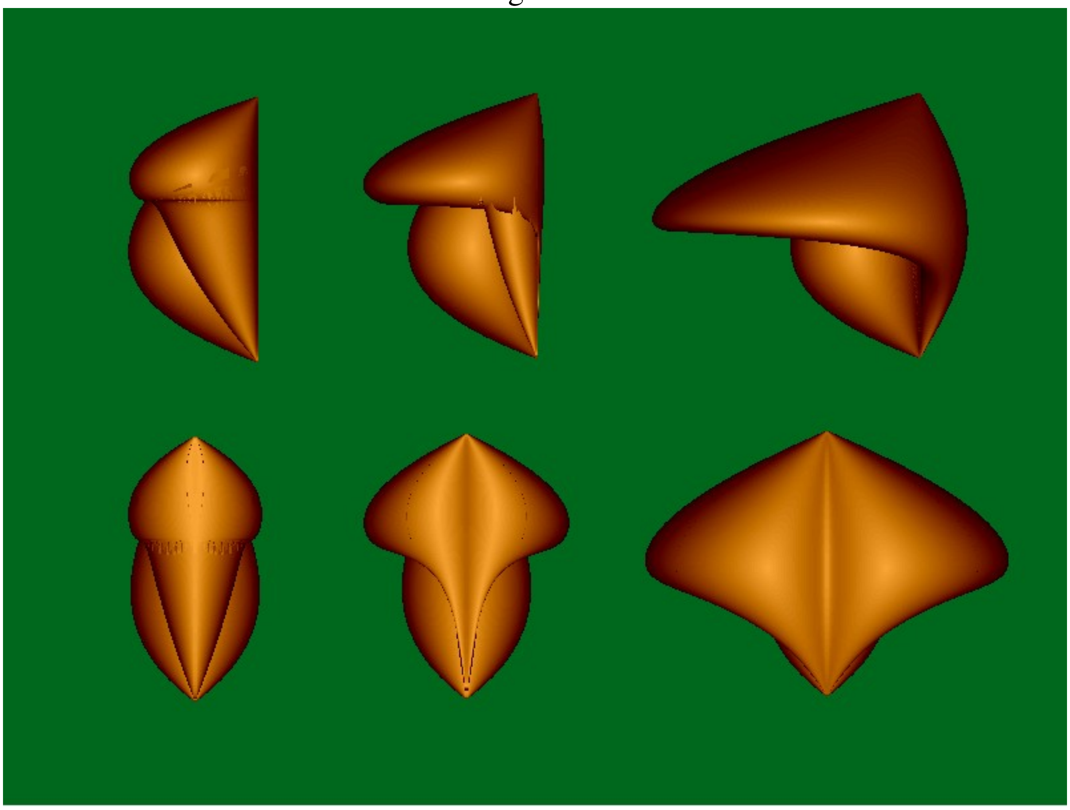
Effect of varying vortex λ with c=0

Figure 5



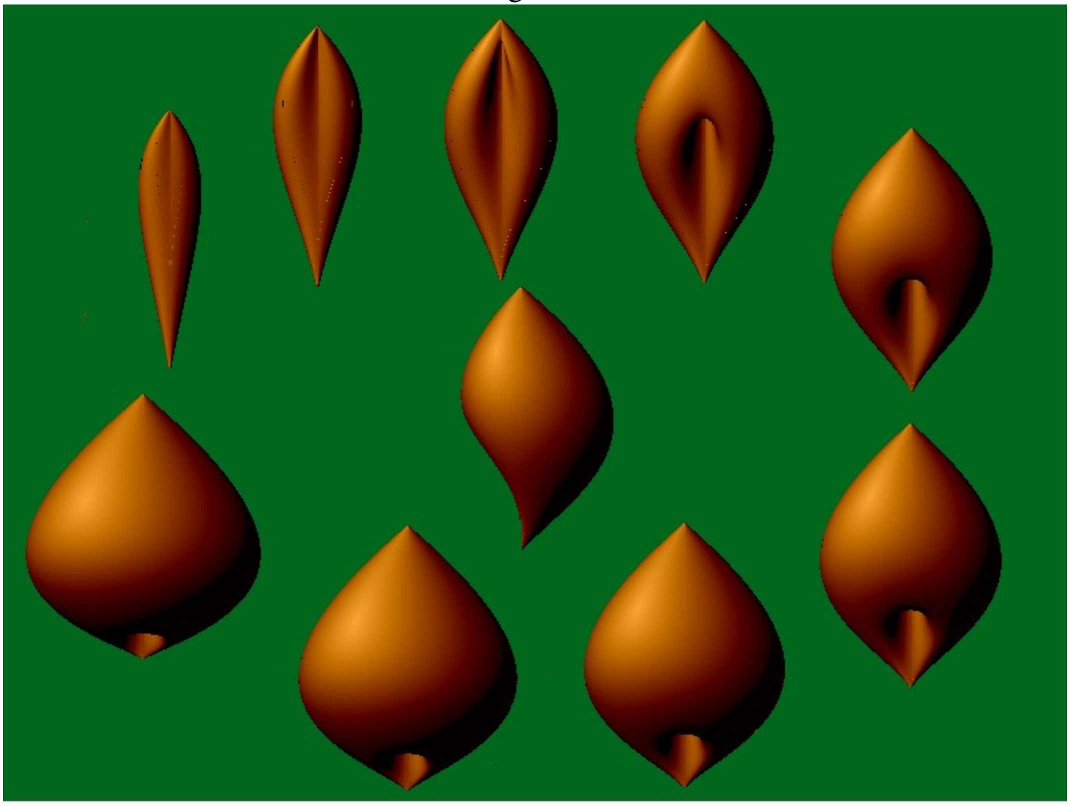
Cases with vortex axis intersecting bud axis

Figure 6



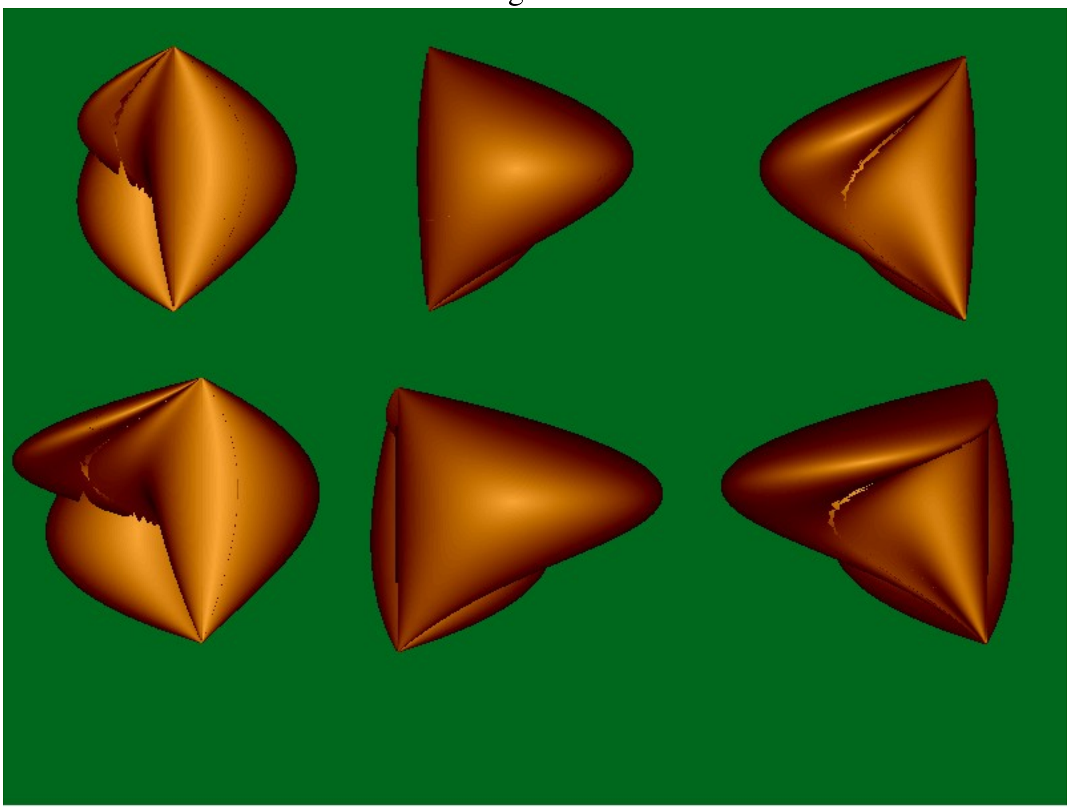
Cases with vortex axis through X

Figure 7



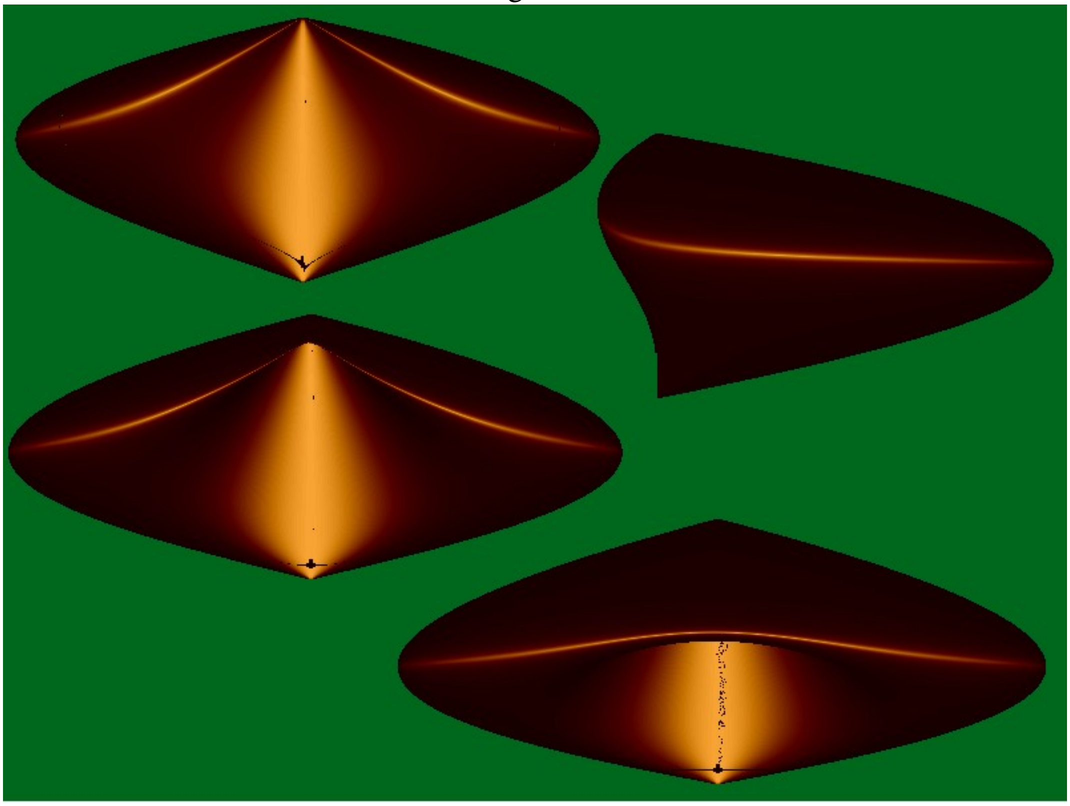
Axis vertical & close to bud axis; λ varying

Figure 8



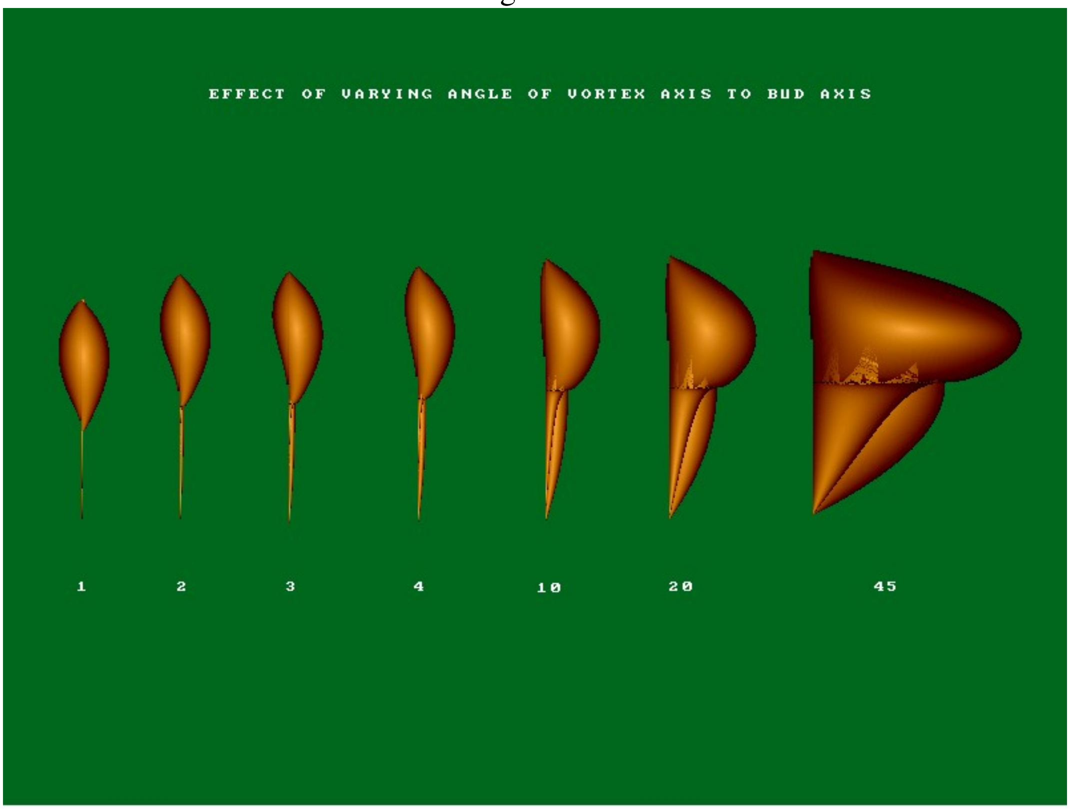
Cases with vortex axis meeting orth line to axis

Figure 9



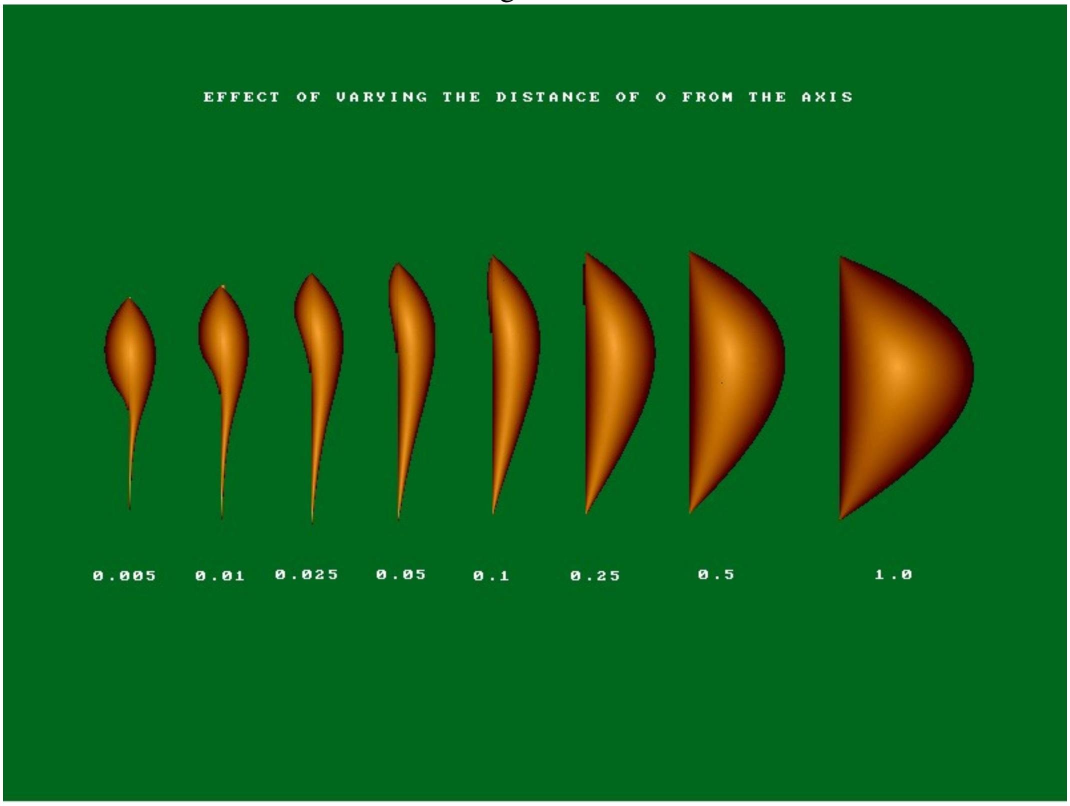
Large d, axis at –40 & at azimuth 160

Figure 10



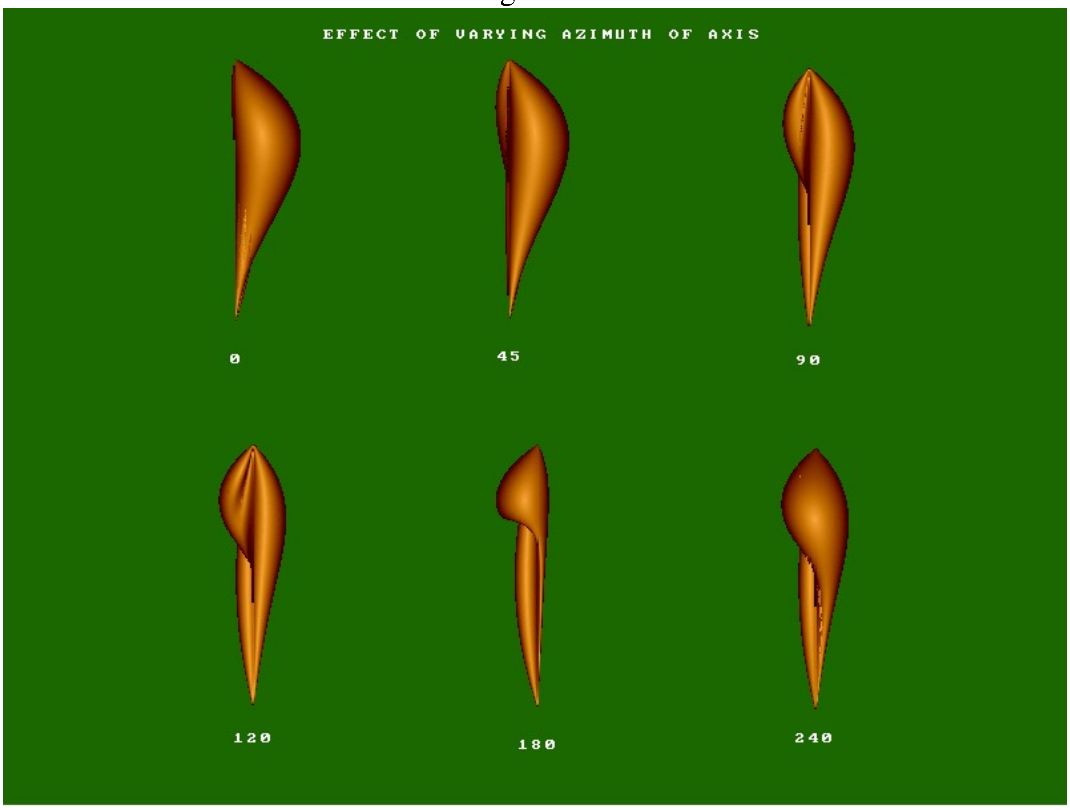
Effect of varying vortex axis angle to vertical

Figure 11



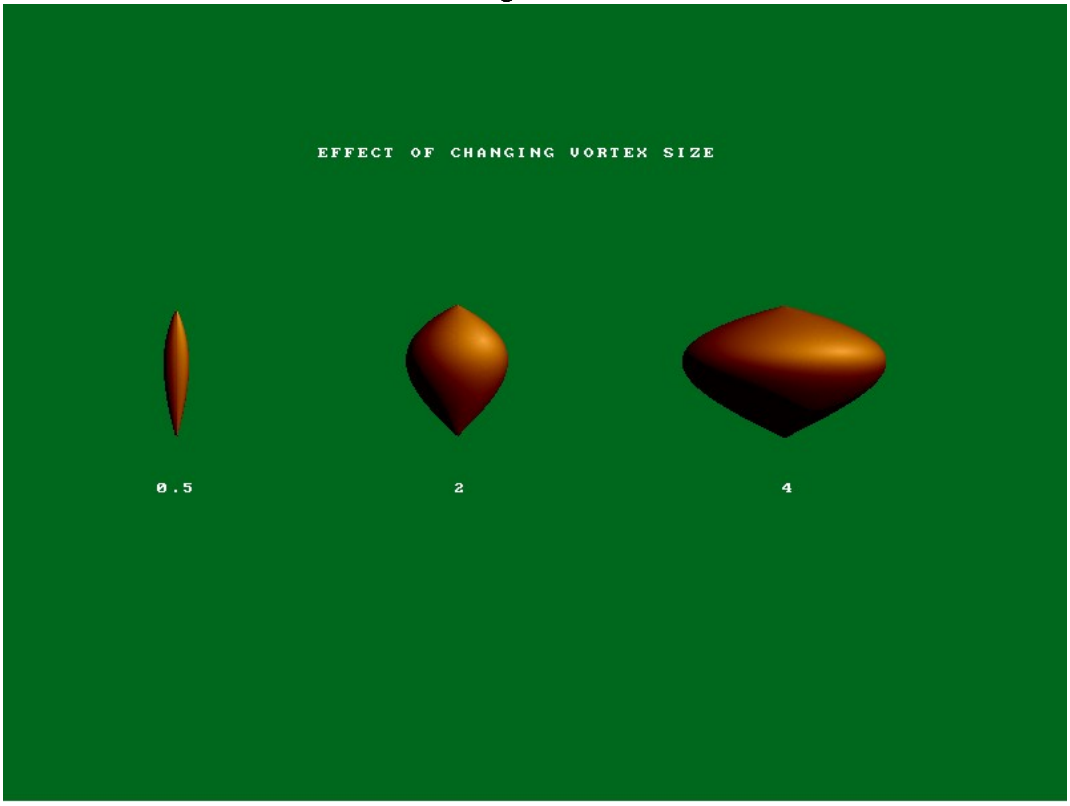
Effext of varying d

Figure 12



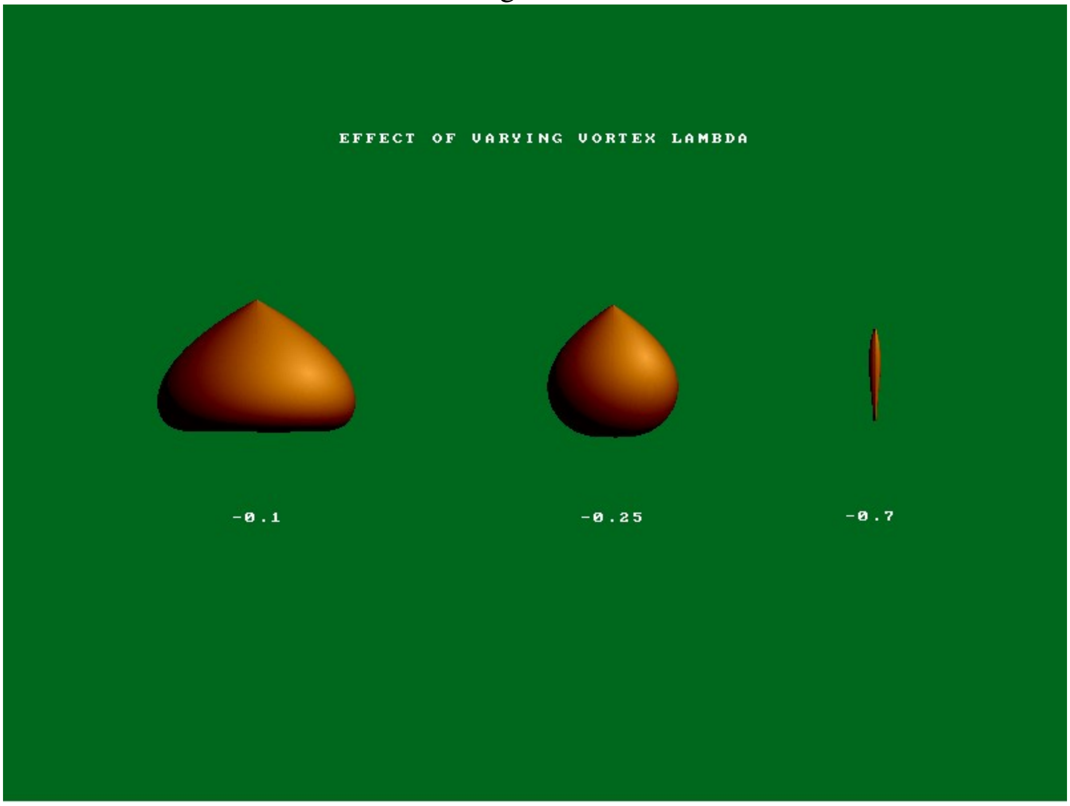
Effect of varying azimuth of vortex axis tilt

Figure 13



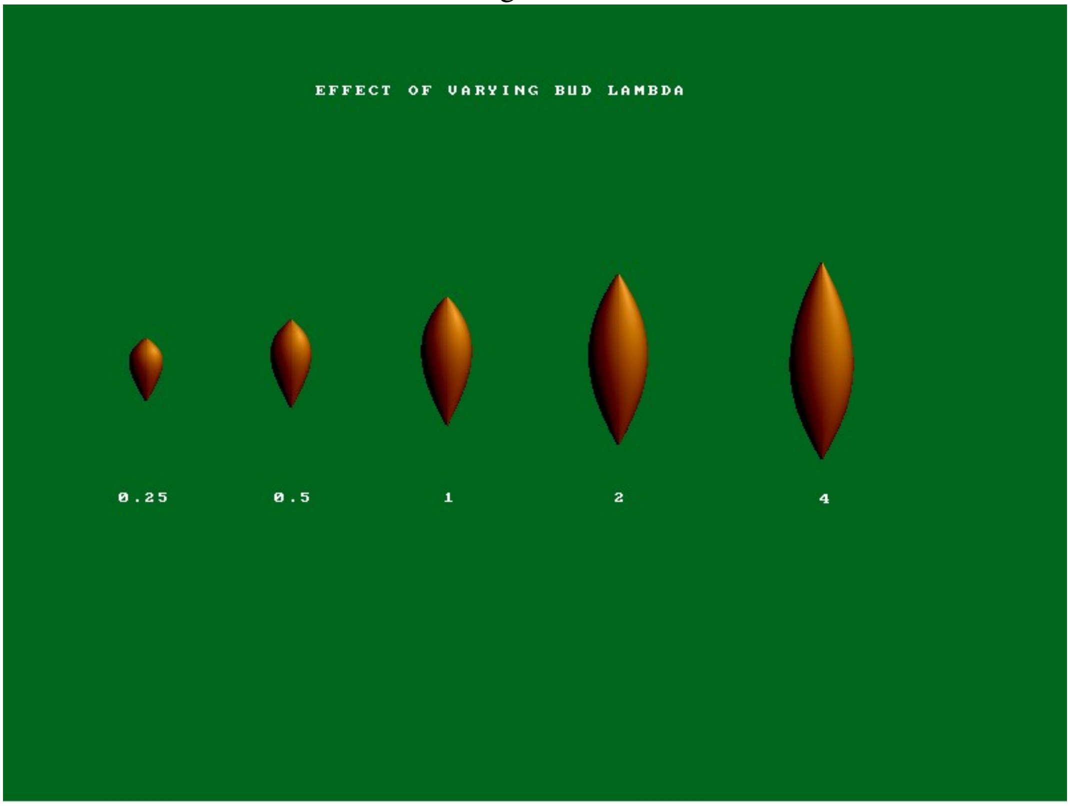
Effect of varying vortex size

Figure 14



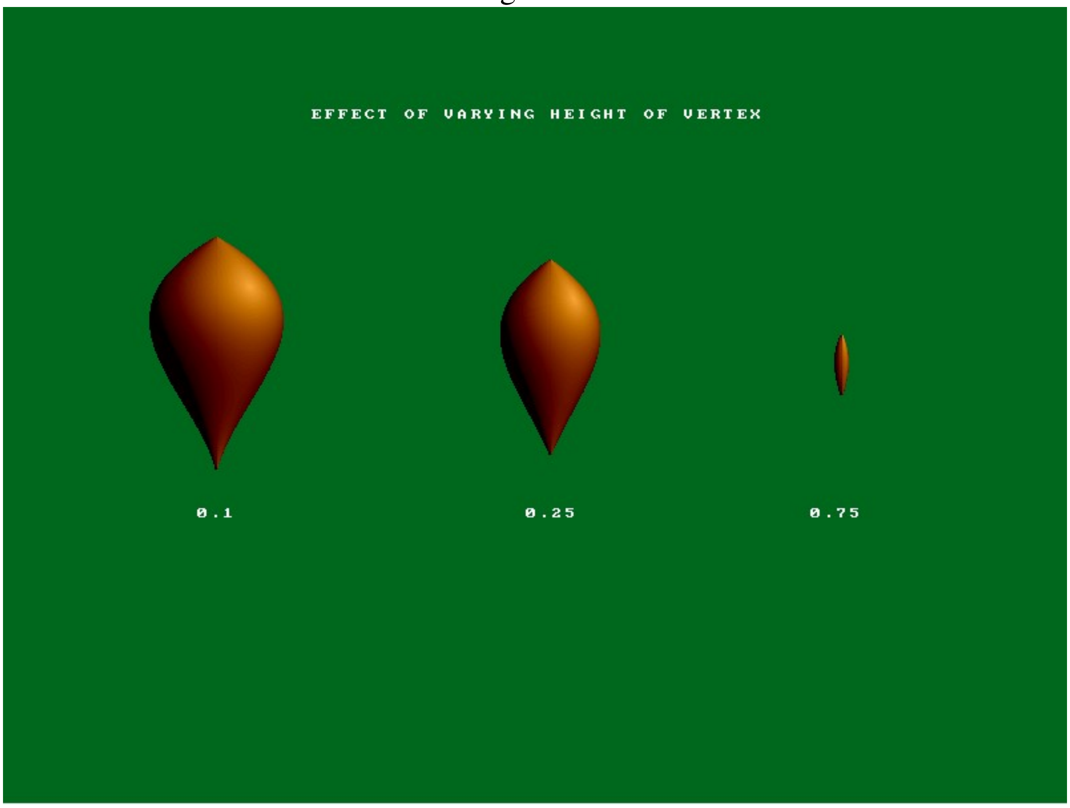
Effect of varying vortex λ

Figure 15



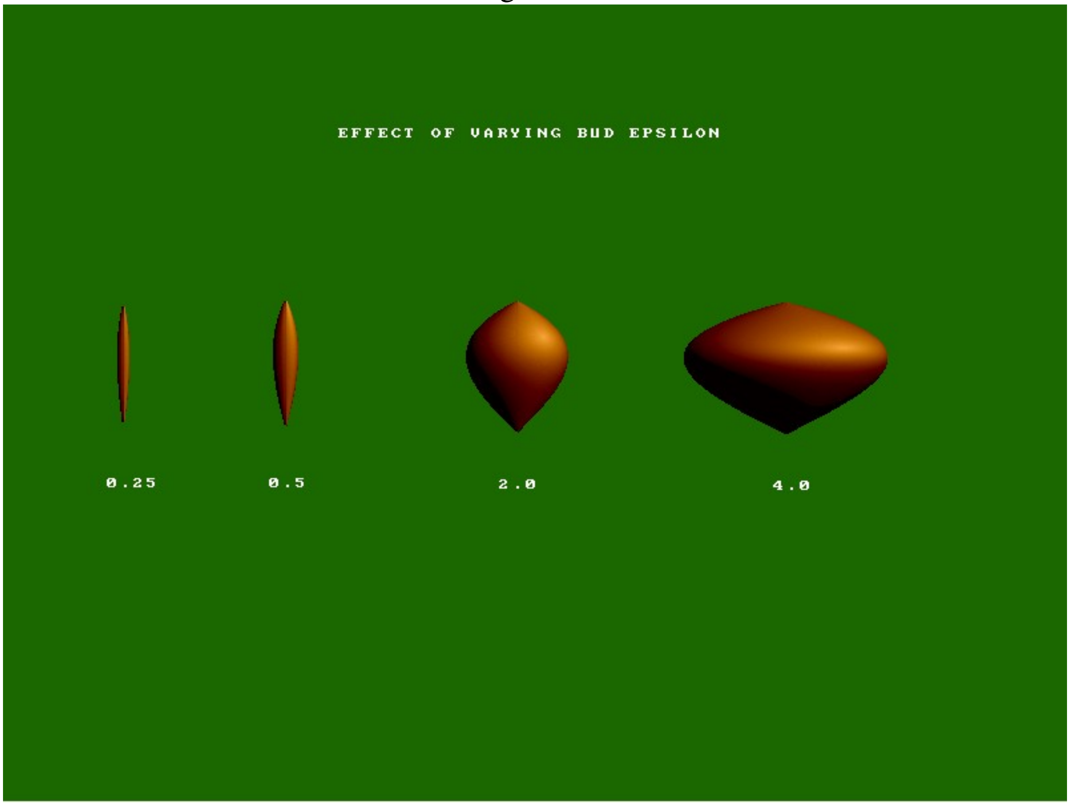
Effect of varying bud λ

Figure 16



Effect of varying fractional height of O

Figure 17



Effect of varying bud ϵ

Elements

The term "elements" is used here in the ancient sense: Fire, Air, Water and Earth corresponding in modern terminology to heat, gas, liquid and solid.   Plasma is yet to be studied in the present context.

GRAVITY and SOLIDS

Given the central thesis then we expect that stresses may appear either in space or counter space or both.   Should they arise only in counter space then they will manifest as forces which are difficult to explain if counter space is not taken into account (as is the case conventionally). A notable example is gravity which Newton never explained, and Einstein also only described. This was the first subject analysed, and it proved possible to obtain Newton's law of gravity, which encouraged further work. It is explained as the gradient of the stress arising from the linkage of points (see Reference 11 for the details). In counter space points are separated by a different kind of measure which is the dual of angle, and is referred to as *shift*. Thus gravity arises from the gradient of shift stress.

The analysis of point linkages has been use to treat gravity, liquids and gases. In each case the gradient of stress arising from point linkages is involved which lends a coherence and consistency to the whole subject. The difference between the states of matter lies in the different kinds of geometry lying behind the linkages:

affine linkages for gases

special affine linkages for liquids

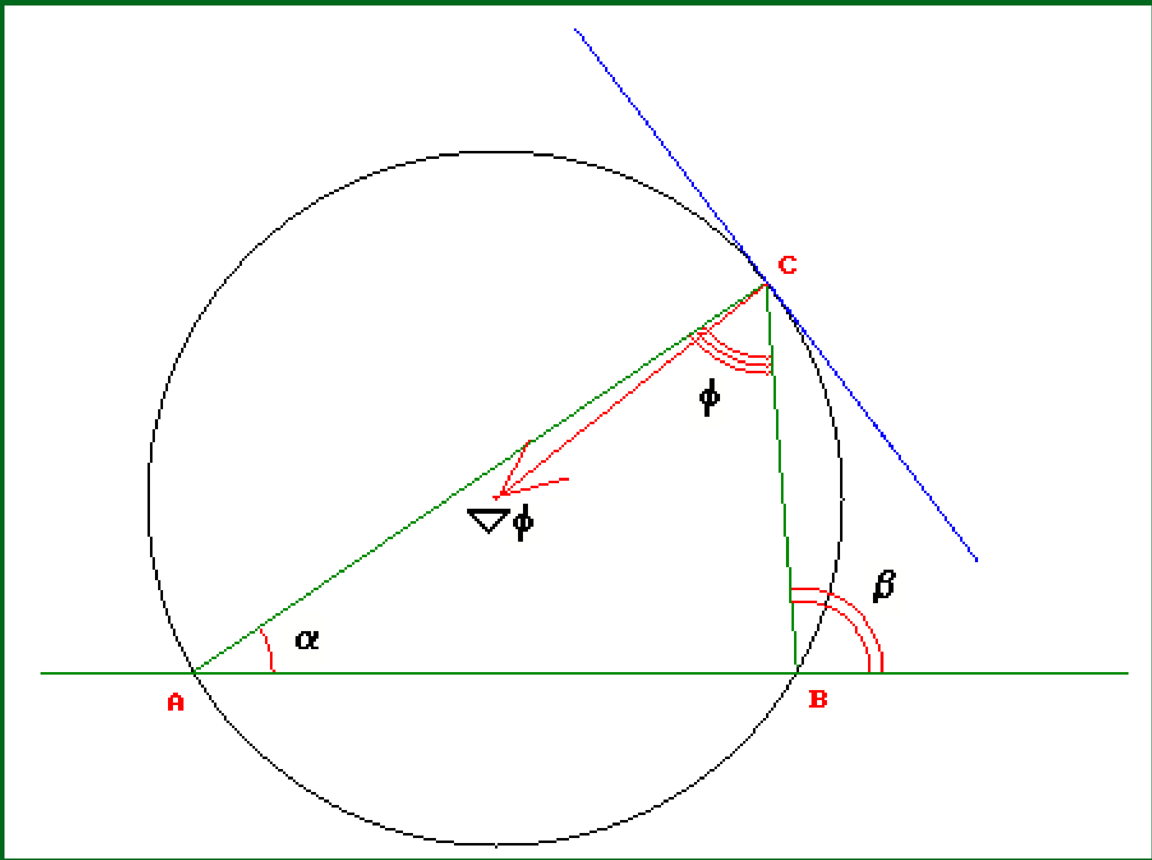
Euclidean metric linkage for solids.

HEAT

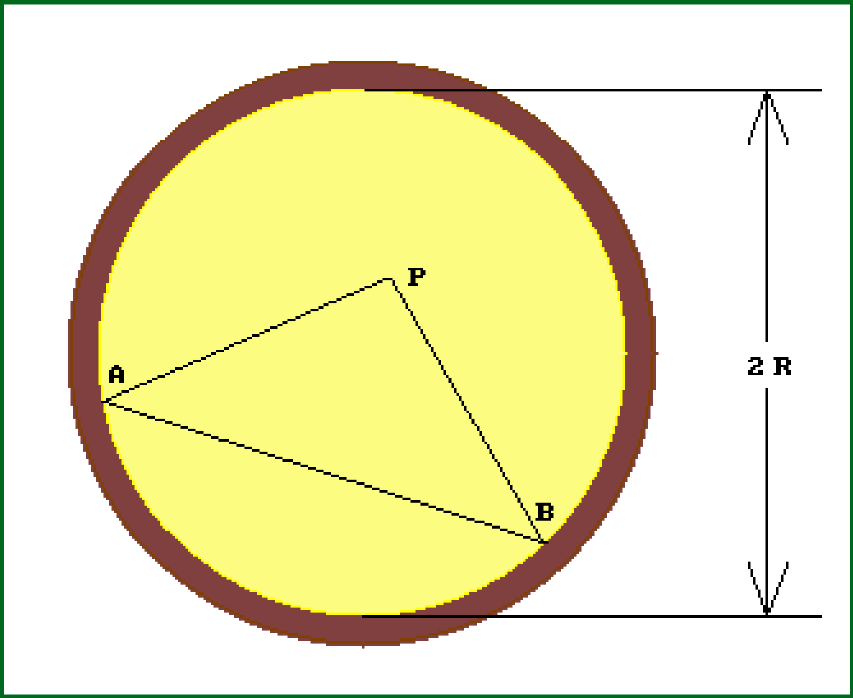
When space and counter space are linked then the calibration or scaling of the two spaces is important. How much shift corresponds to one metre for two points, for example? In the case of planes how is turn scaled to spatial quantities? It has been found that the ideal gas law and the behaviour of liquids is comprehensible if temperature is related to the scaling between the two spaces. This may vary throughout a body in a stochastic manner which gives rise to *scaling strain and stress* which we relate to heat.

GASES

Gases are studied on the basis of an affine linkage between space and counter space. The concept of a point linkage is abstract, and in practice it has proved fruitful to consider a fractal relationship between space and counter space such that every point linkage is a fractal image of the infinitude of the *primal counter space* involved. Different primal counter spaces are envisaged for different elements. This is particularly suited to shift which is a scale-invariant quantity, as fractals are essentially scale-invariant. A quantity of gas is seen as an assemblage of CSIs (counter-space-infinity images) which suffer *affine stress* as each CSI "sees" the others from a different perspective. The linkage here is affine. Hence in the primal counter space there is strain and stress, analysis of which gives the ideal gas law. This is based on the *chord law*:



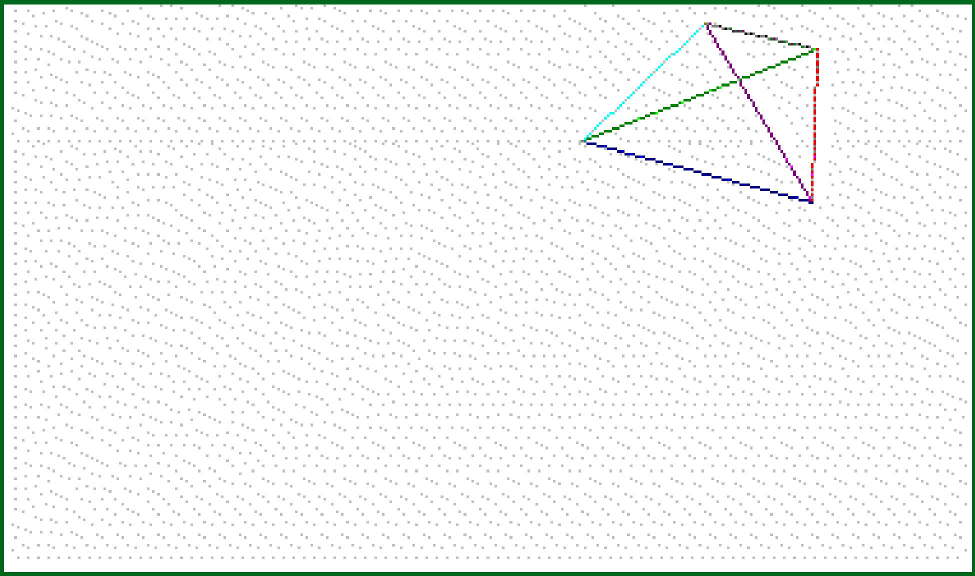
If we have three CSIs at A B and C, the two CSIs at A and B "see" C in conflicting directions denoted by the angles alpha and beta. Their difference phi is a measure of the strain. The gradient of this strain is shown as the red arrow, which must pass through the centre of the circle because the rate of change of phi is zero along the tangent at C. It can be shown (Reference 11) that the magnitude of the gradient at C is proportional to AB/(AC.BC), the actual value depending also upon the scaling. This is used to derive the gas law by summing the stresses for all such triangles, as illustrated below for a metric (solid) container containing an affinely linked gas.



A and B are CSIs anchored in the wall of the container and P is a free one, the chord law being applied to all such triangles for all orientations within the sphere, the nett result being PV=kT where k is a constant depending on the scaling, which thus enables the scaling constant to be found from Boltzmann's constant K.

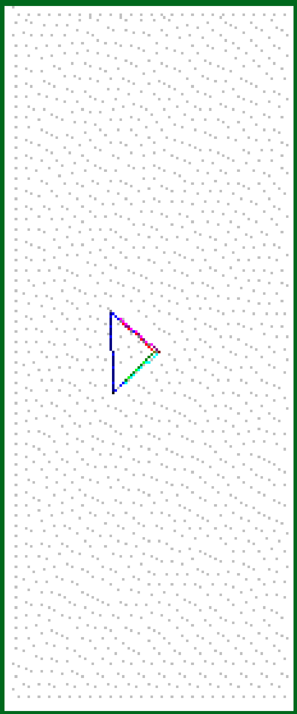
LIQUIDS

Liquids are studied on the basis of a special affine linkage which conserves volume. A constant volume tetrahedron is taken as the basis, just as a triangle was taken for gases. The affine stress gradient is summed vectorially at the vertices, but the result is considerably more complicated than for the chord law. The following animation shows how a tetrahedron released from a particular shape evolves under the action of the stresses (from a computer model of the equations):



The two points to note are: (1) that it stabilises as a regular tetrahedron, and (2) the base to the left moves towards the apex at the upper right. These tetrahedra are only stable when regular, but the equilibrium is dynamic as the residual stresses are not zero, but in a sensitive balance, giving the fluid its sensitivity. The fact that the base moves to the apex (i.e. where the angles are initially greatest) is significant for surface tension and the way a water drop behaves, as tetrahedra within the drop are in equilibrium but those with bases in the surface are not, and strive to pull the surface inwards. Surfaces are thus the principal source of imbalance.

The next animation shows the behaviour of an initially longer, thinner tetrahedron:



Note how the form evolves slowly and rotates until equilibrium sets in quite suddenly (not fully realisable with this animation). This behaviour may refer to vorticity.

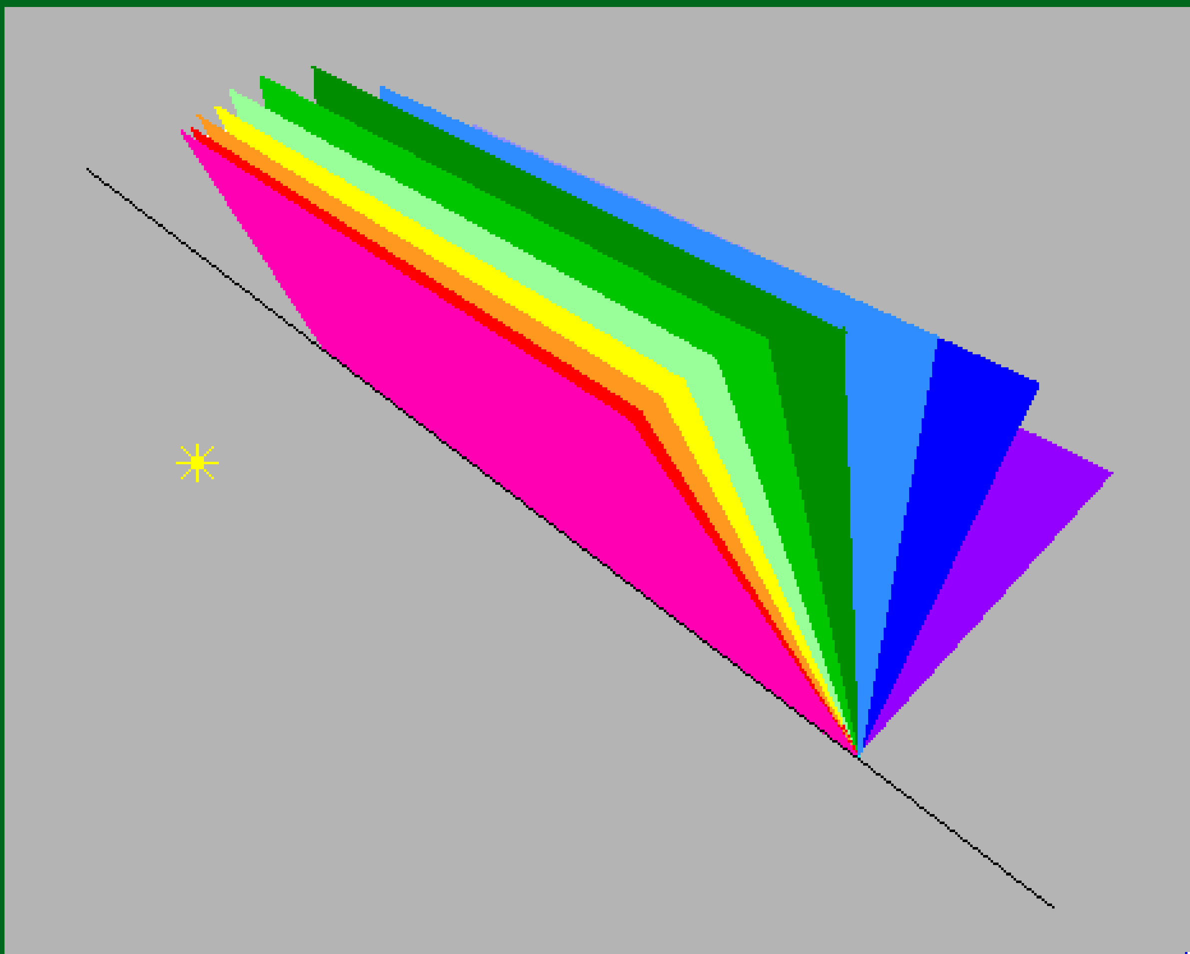
The speed of development in all cases depends upon the scaling between space and counter space (i.e. the temperature).

Flat "tetrahedra" behave chaotically, and have relevance to behaviour in the surface such as Brownian movement and also evaporation. The reason is that they have zero volume and are thus singular for special affine geometry. Again a surface is most significant for such cases.

Thus the behaviour of a volume of liquid is based on the constant volume property of special affine linkages coupled with the action of affine stress.

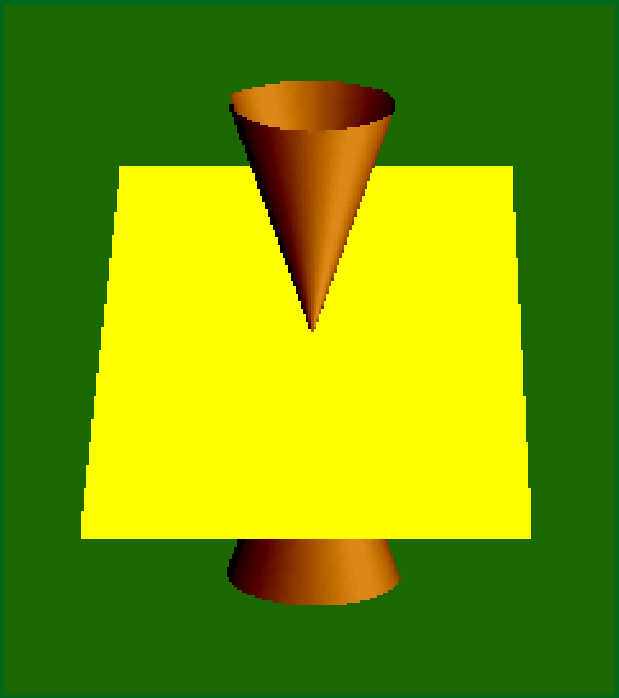
Ethers

TURN

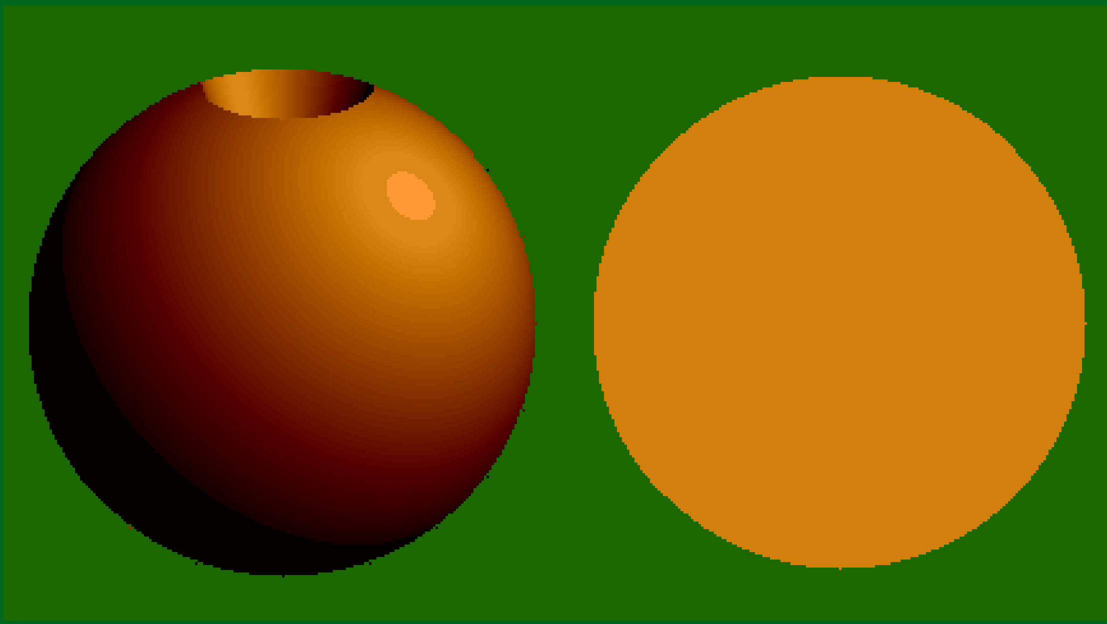


The animation shows how a plane in counter space moves towards its infinity in equal steps for counter space. These are not equal angles, as is obvious, and we refer to this measure as *turn*, which is for counter space the analogue of distance. Its magnitude becomes infinite if the plane reaches the infinitude (shown as a star). The polar relationship between space and counter space means that the measure of separation of planes is polar to that of points in space, while the measure of the separation of points (shift) is polar to that of planes and hence like an angle. Thus pairs of planes can define vectors, but not pairs of points.

This means that the metric of counter space is expressed by turn and shift, whereas that of space is embodied in length and angle. In addition the polar opposite of area and volume may be defined, which are referred to as *polar area* and *polar volume*. The polar area of a cone in counter space is made up of the planes in its vertex, as illustrated below:



Planes in the vertex only have two degrees of freedom and thus make up a polar area, not a polar volume. This may take some getting used to as our Euclidean consciousness tends to regard the region described by the planes as an infinite volume. The polar area is calculated by integration, which is figuratively illustrated in the following animation:



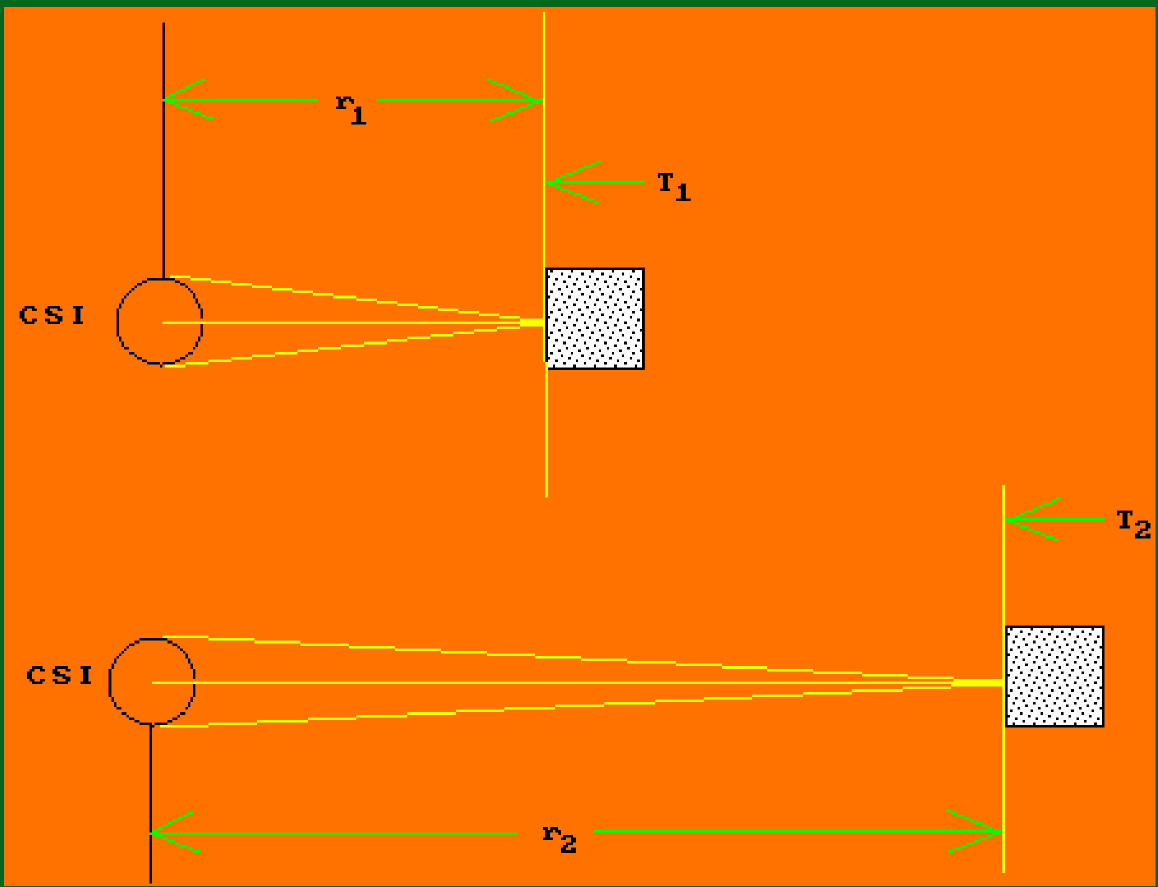
On the left is the integration of the polar area of a cone in counter space and on the right the dual integration of the area of a circle in space. As it is difficult to represent the planes involved in this diagram we have taken successive conical segments bounded by a sphere to represent steps in the progress of the integration. However this is purely representational to help convey the idea, and should not be misunderstood in a point-wise manner. The spherical boundary is adopted to keep the diagram finite, the actual polar area extending outwards to infinity, as do the cones.

The formulae for polar area and volume are the same as those for the dual spatial figures, except that lengths and angles are replaced by turns and shifts respectively (Reference 11).

LIGHT

When dealing with the states of matter we worked with point linkages between space and counter space. For the ethers (as the more subtle aspects of reality are called by Steiner) we are concerned with planar linkages. The most suitable linkage tensor for light is the contravariant bivector, represented by a cone in counter space (dual to the oriented-circle-representation in space). It is suitable for polar affine linkages characteristic of light. This turned out to be an investigation of actual counter space cones acting as photons, the polar area of a photon being constant. Thus photons are initially neither waves nor particles. Their polar area embraces the whole of the apparatus and so the "spooky" multi-path type experiments of modern physics may be more comprehensible. Reflection, refraction, absorption and diffraction are all treated on this basis in Reference 11.

This led to the conclusion that time is the reciprocal of radial turn i.e. the turn between spatially parallel planes. Thus time increases outward from the CSI in counter space. The consequence is that light itself does not in fact have a velocity, but it appears to have one in ordinary space, and moreover that is necessarily constant without the necessity for Relativity. This follows because the product of the radial distance of the apex of a cone from a CSI, and the turn of the orthogonal plane in the apex, is constant. An interaction must occur at the apex (Reference 11), so if the turn is the reciprocal of the time then we have a constant ratio of distance to time, which seems like a velocity for our spatial consciousness. It is independent of the state of motion of the observer.



We see two CSIs emitting photon cones (yellow) interacting at their apices. Since the turn T increases inwards while the radial distance increases outwards, and T is inversely proportional to r, we have r1.T1=r2.T2, so r1/t1=r2/t2=c, a constant which is clearly the so-called velocity of light. The light represented by the polar area is not moving in this way, but an interaction forces the cone to adopt a particular configuration instantaneously, which then gives the appearance of a velocity when interpreted spatially.

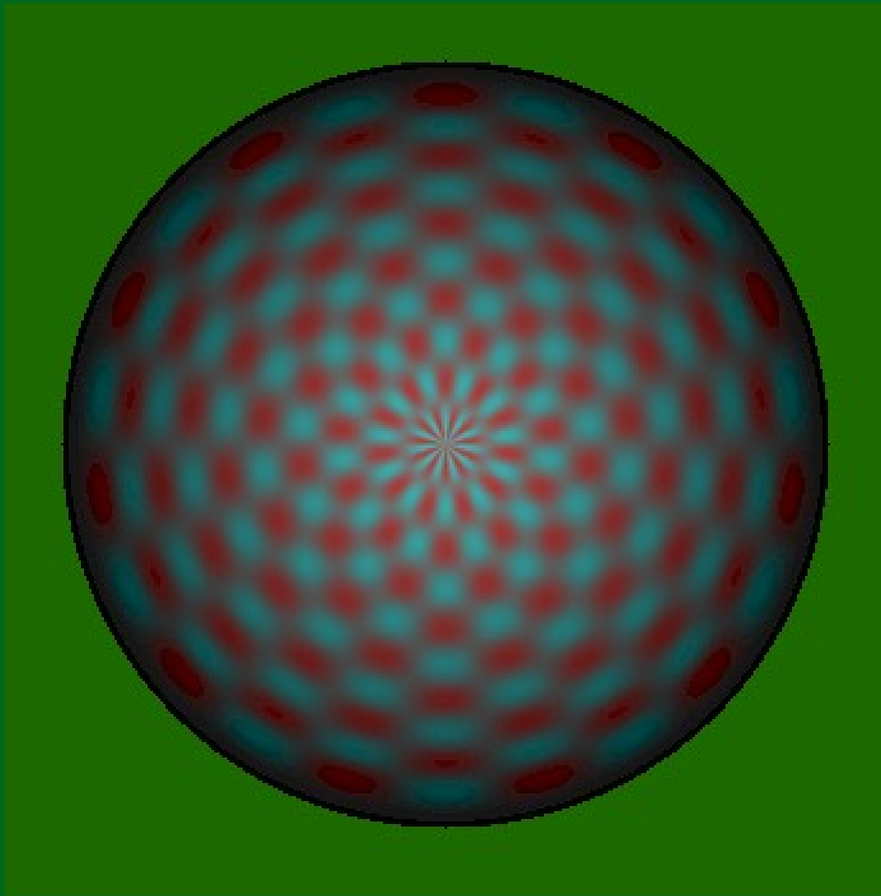
The same view of time arose independently from a consideration of momentum and potential energy.

The residual two-dimensionality is timeless and concerns the ether proper, which need not be linked to space. When the light ether is linked we get photon cones as described above.

This prompted the idea that the ether is concerned with time-invariant processes in counter space, and for light the transformations involved make the polar area of photon cones time-invariant.

CHEMISTRY

An obvious time-invariant field of study is action in the surface of a sphere, noting that this refers to its tangent planes. Surface spherical harmonics provide a suitable approach. They are especially significant when the action is linked to space as then Laplace's equation must be satisfied. They are like standing longitudinal waves in the surface. A standing wave round a circle must consist of a whole number of wavelengths to be single-valued, and a similar restriction exists in the surface of a sphere, but of course in a more complicated manner. The following image shows an example of the distribution or wave pattern for such a harmonic:



This depicts a pointwise distribution for the X(30,11) surface spherical harmonic, red for positive amplitudes and cyan for negative. For counter space the colour of each point represents the magnitude of a surface turn in a plane tangential at that point.

The use of surface spherical harmonics brings in the need for quantisation or whole numbers, and it also brings in rhythm, which suggests that the ether concerned is the chemical ether. This is because Steiner found it could be depicted as the number ether, the tone ether and the chemical ether. Thus the relation of this to chemical action is being explored (Reference 11). In particular chemical bonding is being studied with the help of prolate spheroids. The linkage is polar special affine.

It also suggests that the "waves" of the conventional wave function may be interpreted as rhythms in the chemical ether (i.e. the surface longitudinal waves depicted above), as an alternative to the Born interpretation. It is interesting that the same mathematics is required as for much of quantum physics, but for different reasons. The necessity for time-invariance arises from the whole approach of the work, whereas in conventional physics it is adopted for mathematical convenience.

LIFE

Life ether is concerned with fully metric counter space linkages, which are the most rigid (compared with affine linkages) and are seemingly unsuitable for it. But membranes are fundamental structures in living organisms, being metric in character and yet not rigid. They govern "inside" and "outside" in a most important way e.g. the neuronal membrane which may be interfered with by drugs. Each cell is surrounded by a plasma membrane which governs what may enter or leave.

In counter space the term "inside" apparently has the opposite meaning, for if we consider a sphere with a CSI at its centre then the inside is where no elements making up the polar volume are at infinity for counter space i.e. the planes that do not intersect the sphere. Thus for counter space the "inside" of the sphere is what we would normally call the "outside". This reversal is seen as being significant for membranes, for then all cells of an organism are "inside" the plasma membrane in counter space (for an individual cell). In other words all cells are inside all others!

This is what makes an organism an *organism*.

The synergy of such a system of cells forming an organism in this way will thus govern its growth and healing, and mathematically this kind of synergy can be related to fractals, and may explain why we find fractal forms in Nature. Also the polarities involved invoke path curves which - as explained elsewhere - are ubiquitous in Nature. A particular form of polarity that seems fruitful is the pivot transform.

Algebraic

ALGEBRAIC APPROACH TO COUNTERSPACE

Polariy and Quadric Surfaces

The basic algebra for handling projective geometry is introduced in the Basics page. Important is the duality it so clearly expresses which enables the concept of polarity to be handled conveniently. The equation of a quadric surface is a general homogeneous equation of the second degree in the homogeneous coordinates (x, y, z, w):

$$ax^2 + by^2 + cz^2 + dw^2 + 2exy + 2fxz + 2gxw + 2myw + 2nzw = 0$$

In matrix form this is

$$\begin{bmatrix} x & y & z & w \end{bmatrix} \begin{bmatrix} a & e & f & g \\ e & b & h & m \\ f & h & c & n \\ g & m & n & d \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$$

which may be expressed as $\boldsymbol{x'Qx} = 0$ where $\boldsymbol{x}$ is a column vector, $\boldsymbol{x'}$ is the corresponding row vector and $\boldsymbol{Q}$ is the symmetrical 4x4 matrix representing the quadric surface. Now consider the equation $\boldsymbol{y'Qx} = 0$. We may regard $\boldsymbol{Qx}$ as the coordinates $\boldsymbol{u}$ of a plane so that $\boldsymbol{y'u} = 0$ simply expresses the fact that Y lies in U (using Y to denote the geometric point represented by $\boldsymbol{y}$, etc.). Now Y need not lie on Q, so the expression requires it to lie in the plane U, and any value of $\boldsymbol{x}$ that ensures that satisfies the equation. As X varies in accordance with this we obtain all the planes U in Y. Looking at the situation the other way round, $\boldsymbol{y'Q}$ is also a plane, moreover a fixed plane $\boldsymbol{v}$ since Y is fixed, and as $\boldsymbol{vx} = 0$, all the points X must lie in V. Thus for every fixed point Y the quadric Q determines a plane of points which is the *polar plane* of Y. Conversely given a plane V, $\boldsymbol{v = y'Q}$ for a unique point Y, since $\boldsymbol{y' = Q''v}$ determines Y uniquely, where $\boldsymbol{Q''}$ is the inverse matrix of $\boldsymbol{Q}$. Y is the *pole* of V, and this polar relationship is one-to-one provided the quadric is not singular. If Y lies on Q then one solution for $\boldsymbol{x}$ is $\boldsymbol{x = y}$ as then $\boldsymbol{y'Qy} = 0$ by definition of the quadric, so the plane $\boldsymbol{Qy}$ is the tangent plane at Y since no other point X can satisfy $\boldsymbol{y'Qx} = 0$ and lie on the surface. If Y lies outside the quadric then planes U exist which touch Q in which case, by what we have just seen, X lies on Q (recalling that X is given by $\boldsymbol{u = Qx}$). This corresponds to the diagram in the Basics page where the polar of Y intersects Q when Y lies outside Q, and conversely if Y lies inside Q then no real plane U can be a tangent plane so V does not intersect Q.

A quadric may also be expressed in terms of plane coordinates as follows. In $\boldsymbol{x'Qx} = 0$, $\boldsymbol{u'} = \boldsymbol{x'Q}$ touching at $\boldsymbol{x}$, and $\boldsymbol{u} = \boldsymbol{Qx}$ also touching at $\boldsymbol{x}$. Thus $\boldsymbol{x'Qx} = \boldsymbol{x'QQ''Qx} = \boldsymbol{u'Q''u} = 0$, giving the *class quadric* as the envelope of its tangent planes U, which is the same as the surface described by X. This connection is only valid for non-singular quadrics, but of course any quadric also has a class equation, singular or not (e.g. a cone possesses tangent planes). A slight economy is possible as it is not necessary to divide by the determinant of $\boldsymbol{Q}$ when deriving $\boldsymbol{Q''}$ since we are using homogeneous coordinates, so in the literature we usually find the class quadric expressed as $\boldsymbol{u'(Q)u} = 0$ where $\boldsymbol{(Q)}$ is the adjoint matrix of $\boldsymbol{Q}$.

Projective Classification of Quadrics

There are three distinct types of quadric in purely projective geometry, distinct because no real projective transformation can transform a member of one type into a member of another. By suitable change of coordinates it is possible to reduce the equation of the quadric to *canonical form* (e.g. Reference 8, 9 or 14) where only the terms in the leading diagonal of Q are non-zero. This gives an equation

$$ax^2 + by^2 + cz^2 + dw^2 = 0$$

which is singular if any of a b c or d is zero (cones if one is zero, plane pairs if two are zero, two coincident planes for three zero).

Three distinct possibilities exist for the relative signs of a b c and d:

1. one of opposite sign to the other three,
2. two positive and two negative,
3. all of the same sign.

In the first case, taking d to be negative and reverting to Cartesian coordinates by dividing by w^2, we have $ax^2 + by^2 + cz^2 = d$ which is the equation of an ellipsoid. A similar result is obtained if instead a b or c is negative, recalling that infinity is not invariant so all central quadrics are projectively equivalent.

In the second case, setting $a=A^2$ etc. such that A B C D are all positive, we have for example when b and d are negative the equation (Ax + By)(Ax - By) = (Cz + Dw)(Cz - DW). This is satisfied by any line which is the intersection of the two planes Ax+By-Cz-Dw = 0 and Ax-By-Cz+Dw = 0, and also by the plane pairs Ax+By-Cz+Dw=0, AX-By-Cz-Dw = 0. It is thus a ruled quadric, the two alternatives yielding the two complementary sets of generators.

In the third case the quadric contains no real points and is accordingly purely imaginary.

Cayley's Metric Quadric

We will now briefly outline Cayley's derivation of metric from projective geometry (which followed an initial insight of Laguerre). The problem in projective geometry is that the only numerical invariant is the cross-ratio (of four points, lines or planes), so this is all that is available for use in defining a quantity that is to be thought of as distance or length. Metric geometry is so-called precisely because its legitimate transformations leave lengths and angles invariant, and also areas and volumes. This is untrue in projective geometry. Cayley proposed restricting the allowable projective transformations to those leaving a quadric surface invariant, which is known as the absolute quadric ***G***. Then given two points P and Q, the line PQ intersects ***G*** in two points I and J say. The cross-ratio (PQ,IJ) is now available for the definition of length, as when we make a transformation P and Q move to P' and Q', say, and I and J move to I' and J' such that I' and J' lie on ***G*** since it is invariant (as a whole, note, not pointwise) and the cross-ratio (P'Q',I'J') = (PQ,IJ). Cayley chose the following expression for length:

s = log(PQ, I J)/2i

so that s is indeed invariant. s is imaginary for real ***G*** , and in addition I and J need not be real, so that gives a non-Euclidean geometry. If however ***G*** is an imaginary quadric then I and J are always imaginary, so log(PQ,IJ) is complex and if it is purely imaginary then s is real. If we select the singular imaginary disk quadric at infinity given by x^2 + y^2 + z^2 = 0 = w then we recover the familiar Pythagorean expression for length, which is of course why the above expression was selected by Cayley. This is rather messy and limiting arguments must be used. The clearest exposition is given in Reference 15 for two dimensions. The result is readily generalised to three dimensions giving for Euclidean geometry the length s between ***x*** and ***y*** as:

$$s^2 = \frac{(x_0y_3 - y_0x_3)^2 + (x_1y_3 - y_1x_3)^2 + (x_2y_3 - y_2x_3)^2}{x_3^2y_3^2}$$

which reduces to Pythagoras' Theorem for Cartesian coordinates with x3 = y3 = 1.

For the angle between two planes U and V Cayley took the two planes I and J in the line (U,V) which are tangential to ***G***, and then used

cos(theta) = log(UV, I J)/2i

For the Euclidean ***G*** the angle between ***u*** and ***v*** in terms of their plane coordinates is then

$$\cos\theta = \frac{u_0v_0 + u_1v_1 + u_2v_2}{\sqrt{\left(u_0^2 + u_1^2 + u_2^2\right)\left(v_0^2 + v_1^2 + v_2^2\right)}}$$

which is the familiar normalised inner product for the cosine.

Thus choosing an imaginary circle in the plane at infinity gives the Euclidean metric. A circle may be regarded as a singular class quadric known as a *disk quadric*, in the sense that there are an infinite number of axial pencils in its tangents which may be thought of as its tangent planes. Those planes are of course imaginary in the present case.

Counterspace Metric

We may dualise a disc quadric as follows: we have the dual of the plane of the circle as a point O, the duals of its tangents as lines in O forming a cone, and the duals of its "tangent planes" (axial pencils in the tangents) as the points of the lines in O i.e. we simply have a cone of points, which is of course more readily felt to be a quadric! For an imaginary circle O is still real (as polar of the real plane at infinity), but its tangents and "tangent planes" are imaginary, so the cone is imaginary apart from its real vertex. It is however a *wrap* of tangent planes, and therefore a class quadric, since it is dual to ***G*** which is treated as being composed of imaginary points. George Adams (Reference 5) suggested using this quadric, say ***H***, as the absolute quadric defining the metric for a quite different kind of space. It is different from the usual notion of non-Euclidean geometry in two main ways: first of all it is based on a class quadric, and secondly that implies the fundamental metric relates planes rather than points. In Relativity the metric tensor ***g*** determines how infinitesimal coordinate displacements may be related to the corresponding infinitesimal distance displacements. This is necessary for general coordinates e.g. even for ordinary spherical polar coordinates. Now ***g*** is a symmetrical matrix and thus may also be regarded as a quadric, which is exactly the connection between Cayley's work and the metric tensor. Indeed a grasp of Cayley's work gives an immediate intuitive feel for the metric tensor. For a curved space the components of ***g*** are functions of the coordinates, which obey special conditions to ensure the matrix is also a tensor, and thus we can visualise the absolute quadric varying from point to point in a curved space, which is all that is meant by the forbidding formalism of the metric tensor. An important point is that the so-called *signature* of the quadric cannot change. This simply means that no real transformation can change its type e.g. from any of an ellipsoid, hyperboloid, paraboloid, ruled quadric or imaginary quadric to another of them.

Adam's ***H*** fully expresses the metric of a new kind of space, but noting that ***g*** is always assumed to define the distance between points whereas ***H*** defines a new kind of displacement between planes that is not an angle. We discover this by dualising the above expression for distance in ordinary space, giving the displacement between two planes ***u*** and ***v*** as:

$$\tau^2 = \frac{(u_0 v_3 - v_0 u_3)^2 + (u_1 v_3 - v_1 u_3)^2 + (u_2 v_3 - v_2 u_3)^2}{u_3^2 v_3^2}$$

We will refer to tau as the *turn* between U and V. That it is not an angle is clear from the fact that it may become infinite if u3 or v3 is zero. It is fully analogous to distance in that sense, but refers to planes. Adams studied it by means of projective measures.

The geometry characterised in this way is technically *polar-Euclidean geometry* as is clear from its derivation, and it is usually referred to as *counterspace* when thought of as the geometry of another kind of space.

So far we have not said anything about the location of O. It acts as infinity for counterspace, being the dual of the plane at infinity, but the process of dualising does not otherwise locate it. Indeed we are free to locate this real point anywhere in our ordinary space, and in so doing we establish a *linkage* between the two spaces. Lacking linkages the the two spaces are quite disjoint. We refer to such a linkage as a *CSI* (counterspace infinity). The turn tau becomes infinite if either of the planes contains O i.e. is "at infinity".

We can also dualise the above expression for the angle between two planes in space to give the separation between two points in counterspace, which we will call *shift*:

$$\cos\sigma = \frac{x_0 y_0 + x_1 y_1 + x_2 y_2}{\sqrt{\left(x_0^2 + x_1^2 + x_2^2\right)\left(y_0^2 + y_1^2 + y_2^2\right)}}$$

Thus sigma behaves just like an angle, which is to be expected from the dualising process. In other words points are separated in counterspace by a quantity which is never infinite, a notion that takes some getting used to. We may have *parallel points* in counterspace, but not parallel planes. Thus if the vectors ***x*** and ***y*** are parallel but the points are distinct then sigma is zero, the dual of two distinct parallel planes. A useful "crutch" is to see that the numerical value of sigma equals that of the angle between the lines XO and YO in space, regarding ***x*** and ***y*** as position vectors wrt O. However, ***x*** and ***y*** are shift coordinates, not distance coordinates, and such a visualisation is only valid if the points are linked.

Thus far we have treated counterspace as a "flat" space since its metric ***H*** does not vary with position.

Up Home

Algebra

Projective Geometry may also be studied by means of algebra. Linear transformations are expressed as matrices, and the transformation of a point or plane is accomplished by multiplying the vector representing it by the transformation matrix.

HOMOGENEOUS COORDINATES

The Cartesian coordinates of a point may be expressed as (x,y,z) with respect to the three orthogonal axes. The problem encountered in using them, however, is that ideal points at infinity cannot be handled because x,y or z (or all three) become infinite. If a point moves towards infinity in a fixed direction then the ratios x : y : z remain constant. We may introduce a fourth number w and re-express the coordinates as x/w : y/w : z/w, noting that the ratios are unaffected. If we multiply all coordinates by a constant k the ratios are still unaffected. We now re-express the point as (x,y,z,w) as if we were working in four dimensions i.e. we regard w as a fourth coordinate. If w becomes zero then we see that x/w, y/w and z/w each become infinite to give us a point at infinity, but instead of retaining these improper ratios we instead express that fact as (x,y,z,0). This formulation retains intact the ratios of x : y : z of the point before it reached infinity, and we use w=0 to indicate we have gone to infinity. Thus for each direction in space (x,y,z,0) is unique, the twofold infinity of ratios x : y : z representing that direction and (x,y,z,0) its ideal point. Two aspects should be noted:

1. (x,y,z,1) returns us to the Cartesian coordinates when w=1 is discarded;

2. (kx,ky,kz,kw) is the same point as (x,y,z,w) as we are now only interested in ratios.

These coordinates are known as *homogeneous coordinates* because they still refer to three dimensions despite the use of four coordinates, and the coordinates are homogeneous in the sense that they are are not absolute but enter into equations fully symmetrically, just as a homogeneous equation contains all products of its variables to a fixed overall power.

(x,0,0,0) is the point at infinity on the x-axis, and similarly for the y and z axes. Since we may divide throughout by k=x this simplifies to (1,0,0,0).

(0,0,0,1) is the origin.

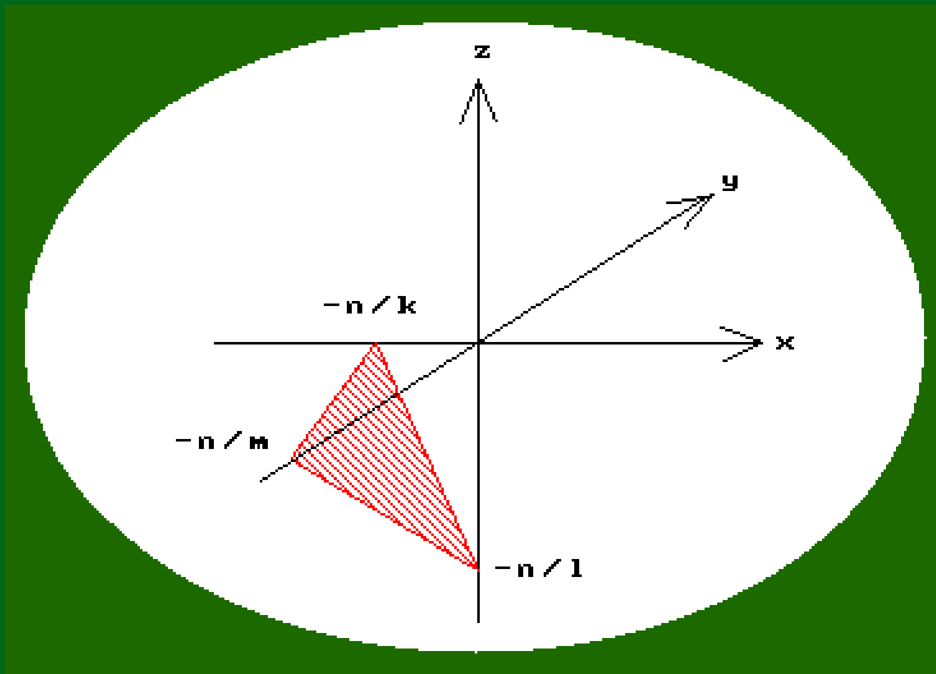
Once we switch to homogeneous coordinates the axes need not remain orthogonal, and we end up with a *tetrahedron of reference* with vertices (1,0,0,0), (0,1,0,0), (0,0,1,0) and (0,0,0,1). All connection with Cartesian coordinates is then lost as distances can no longer be associated with x,y,z and w. This expresses the non-metric nature of projective geometry. The infinite plane is no longer defined as the plane w=0 but can be any face of the tetrahedron, consistent with the fact that an infinite plane is not defined for projective geometry, only for affine and metric geometry.

DUALITY

If we take y=0 then we have all the points in the XZW plane. If we take x+y=0 then we have all the points in the plane for which x=-y. Generally a linear equation in x,y,z,w yields a plane i.e.

$$kx + ly + mz + nw = 0$$

for constant k,l,m,n is the equation of a plane. Now suppose we hold x,y,z,w constant and vary k,l,m,n while satisfying the equation. Clearly we obtain all possible quadruples (k,l,m,n) satisfying the equation for that fixed point (x,y,x,w) i.e. all possible planes containing (x,y,x,w), from which it is clear that (k,l,m,n) may be regarded as the coordinates of the planes. The duality of point and plane is beautifully expressed by the symmetry of the equation. The meaning of these coordinates may be appreciated if we think of the Cartesian special case with w=1. On the x-axis y=z=0 so x=-n/k, and similarly on the y- and z-axes, so the plane represented by the coordinates is as illustrated below.



TRANSFORMATIONS

A linear transformation (x',y',z',w') = f(x,y,z,w) is such that x y z w enter the function f homogeneously to the first power. This means we cannot have terms such as x^2, xy, yw or xyz for example. Thus x' = ax+by+cz+dw for some constants a b c d, and similarly for y', z' and w'. The most convenient way of collecting together these linear equations for x' y' z' and w' is to express them in matrix form:

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ j & k & l & m \\ n & p & q & r \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

recalling that the inner product of a row of the square matrix with the right hand column vector gives the corresponding term in the left hand column vector. We may denote this also as ***x'=Ax*** where capitals denote matrices and lower case letters represent column vectors.

The same transformation may be applied to a plane (s,t,u,v) :

$$\begin{bmatrix} s' \\ t' \\ u' \\ v' \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ j & k & l & m \\ n & p & q & r \end{bmatrix} \begin{bmatrix} s \\ t \\ u \\ v \end{bmatrix}$$

It has long been known that projective geometry may be expressed in terms of linear transformations such as these. However the actual geometry is easily lost sight of if we are not careful !

Generally the point ***x'*** is distinct from ***x***, but we may ask if there are any points that correspond to themselves. If so then such a point ***p*** is such that ***p=Tp***. Because we are concerned with ratios rather than absolute values it is more accurate to set k***p=Tp*** for some constant k. To solve this for ***p*** we need to multiply the left hand side by the unit matrix ***I*** (which has 1 in the leading diagonal and 0 elsewhere e.g. in the above square matrix that would mean a=f=l=r=1 and b=c=d= ... =q=0). Then we have the equation (***T***-k***I***)***p***=0, and as we do not want ***p***=0 then ***T***-k***I***=0. This really consists of four simultaneous equations in k which give rise to the *characteristic equation* which is a fourth order equation in k. The four roots are known as the *characteristic values* or *eigenvalues* and from them it is possible to derive four vectors ***p*** which transform into themselves i.e. in geometric parlance there are four invariant points. There can be no more than four provided the roots are distinct, and furthermore they need not be real. Applying the same idea to find the invariant planes ***u*** gives (***T***-k***I***)***u***=0 and hence the very same equation ***T***-k***I***=0. The four planes must evidently each contain three of the invariant points as such a triple determines an invariant plane. The nett result is an invariant tetrahedron with four invariant vertices, four invariant planes and six invariant edges. It is non-degenerate if the four characteristic roots are real and distinct as then the invariant points are also real and distinct. If however some of the roots are complex the tetrahedron possesses imaginary elements. In particular the so-called *semi-imaginary* tetrahedron arises when two of the roots are complex conjugates, as then only two real invariant planes, points and lines arise. Pairs of equal real roots give rise to lines of invariant points (which are also the axes of axial pencils of invariant planes). This is all described for example in Reference 14.

PATH CURVES

Denoting a square transformation matrix such as that above by ***T***, a path curve arises when it is repeatedly applied to an initial point. If that is ***a*** then a new point ***b***=***Ta*** arises. We now apply the same transformation to ***b*** to give ***c***=***Tb***=***TTa*** and so on. Continuing in this way the series of points ***a b c ...*** are found to lie on a curve. The nature of the curve depends upon the characteristic roots of the matrix ***T***. If all four are real and distinct then there are four invariant points as we have just seen, and the curve passes from one to another of them. If two are conjugate imaginary then the egg and vortex spirals arise that are described on the Path Curve page.

Felix Klein discovered path curves and Sophus Lie gave the transformation for a continuous curve in place of the discrete recursive approach above which places points on that curve.

Instead we may start with an initial plane ***u*** and transform it to ***v=Tu*** and so on. This results in the polar of a locus which is called a *developable*, consisting of a single-parameter sequence of planes that are the osculating planes to the path curve, but the latter is now more strictly referred to as the *cuspidal edge* of the developable. An osculating plane has triple contact with the curve.

Up Home

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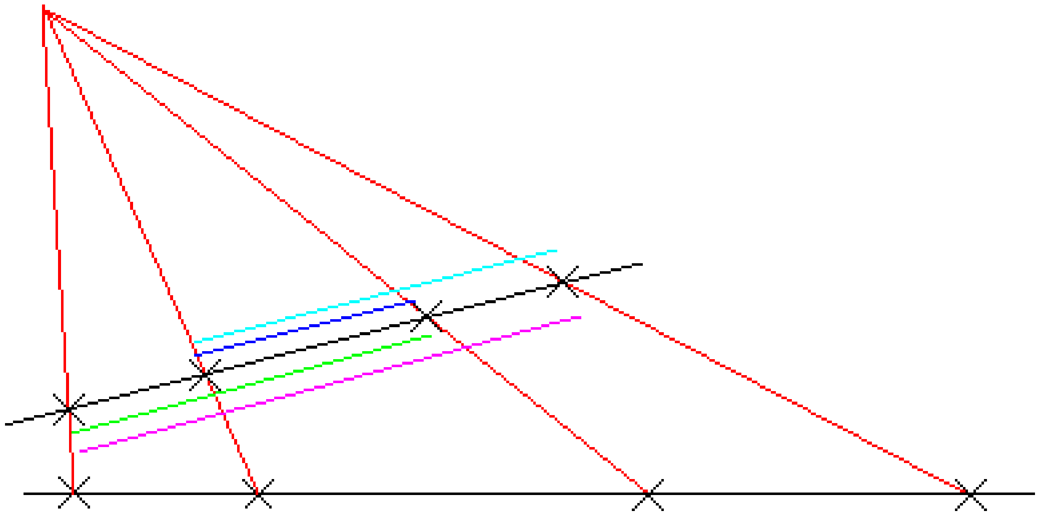

$$\frac{\dfrac{215}{337}}{\dfrac{146}{267}} = 1{\cdot}17 = \frac{\dfrac{141}{193}}{\dfrac{86}{138}}$$

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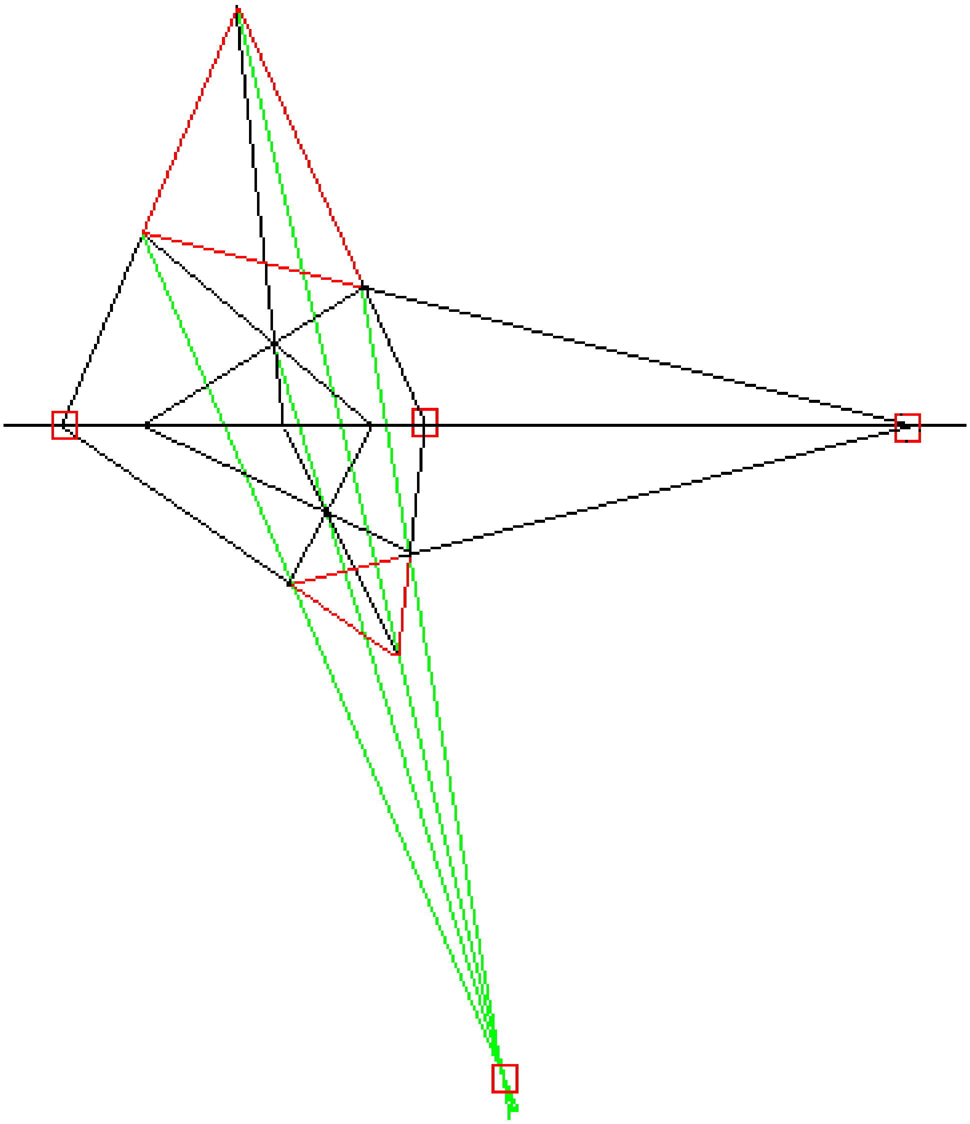

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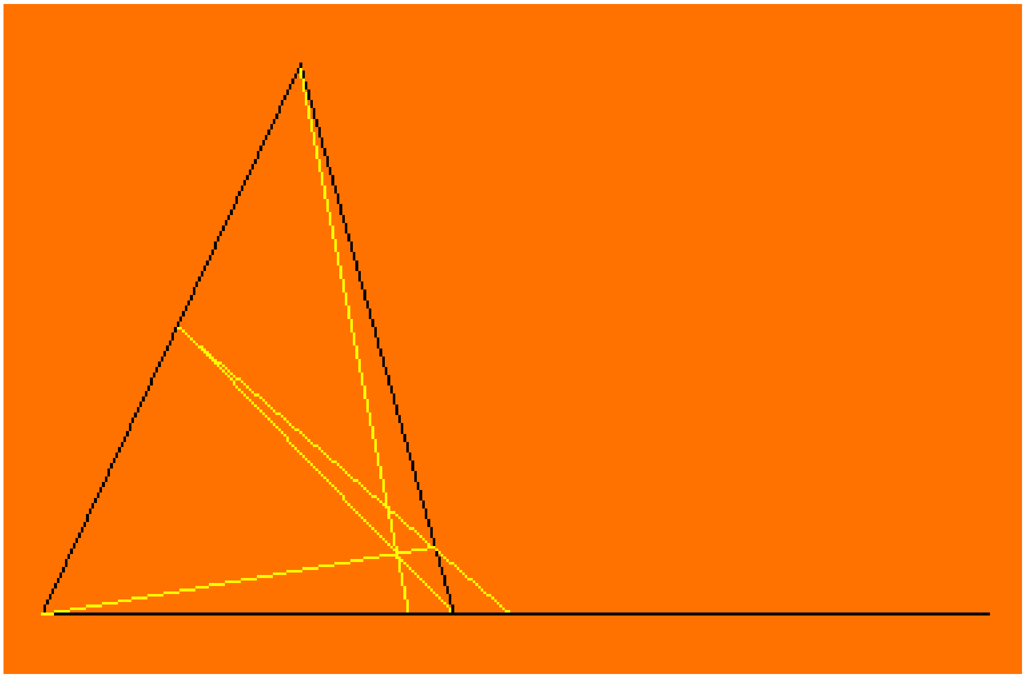

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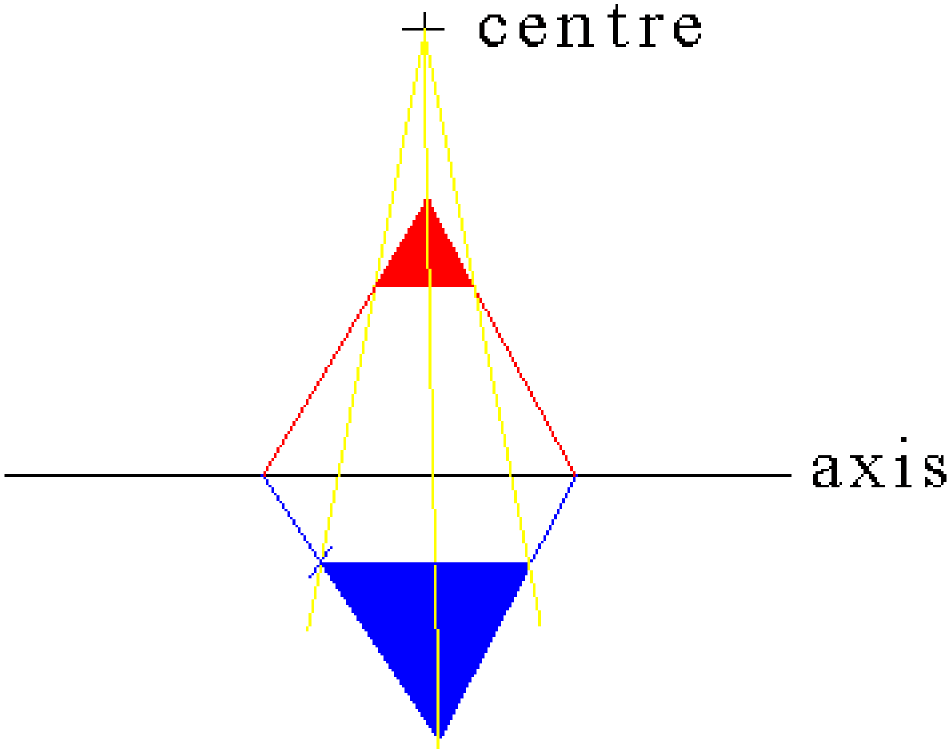

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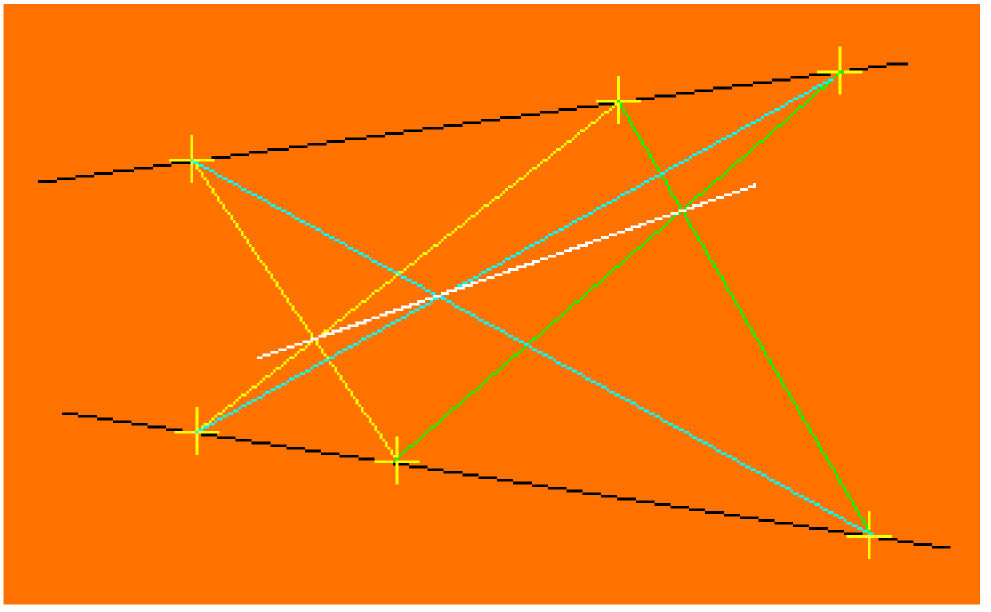

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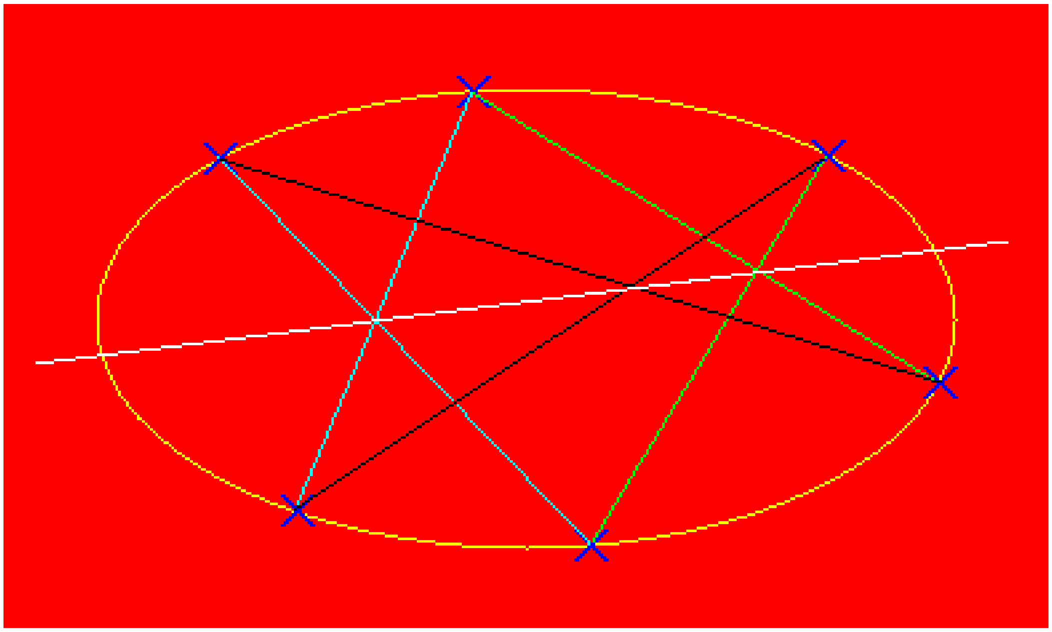

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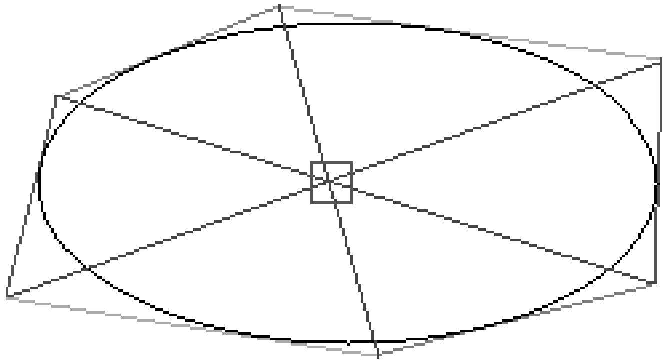

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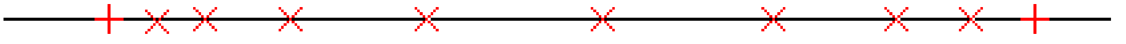

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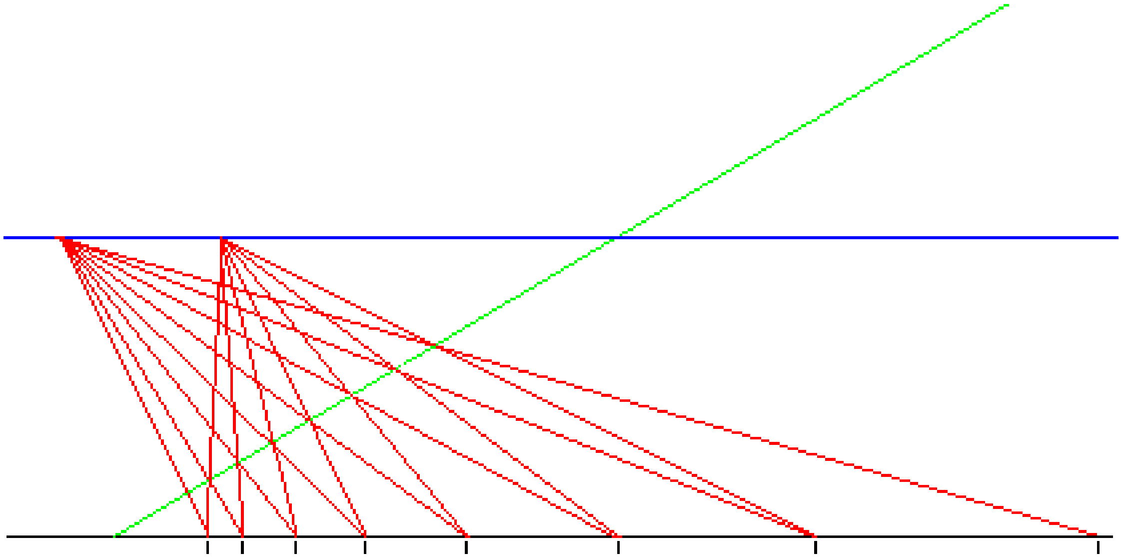

geometric series

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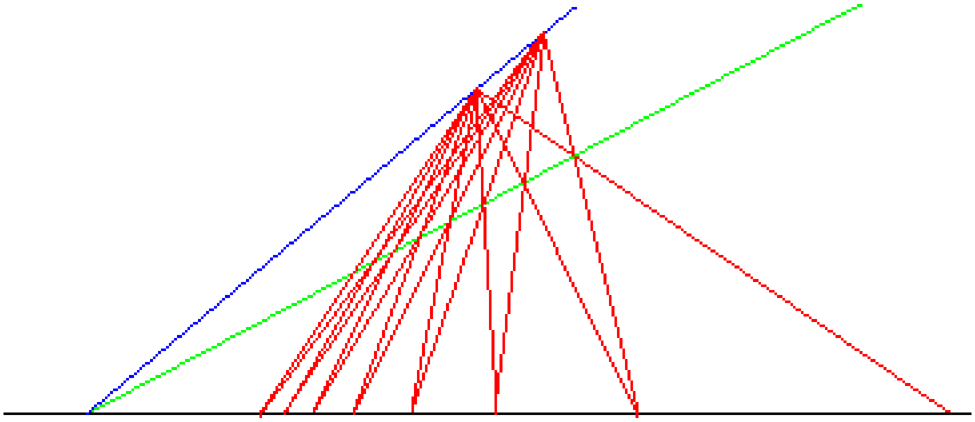

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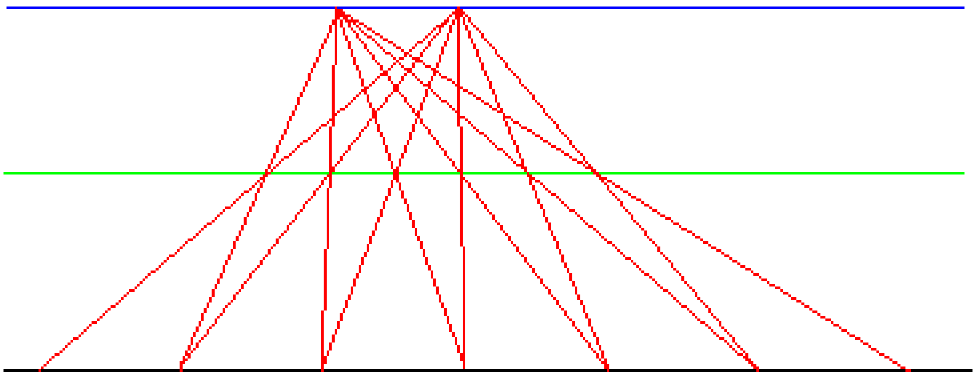

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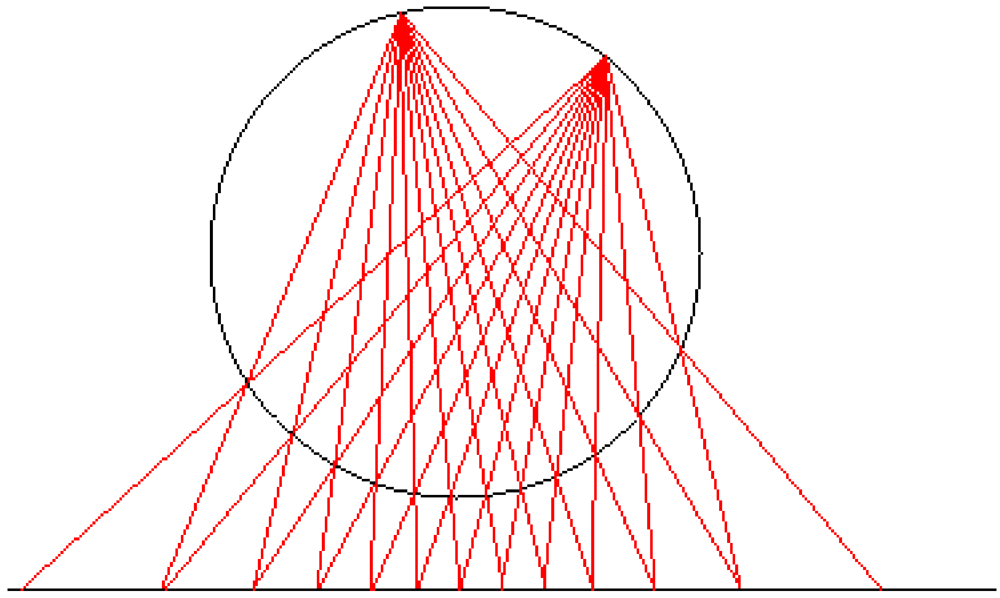

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